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Multigroup Transport Theory with Anisotropic Scattering

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The special case of a triangular transfer matrix relevant to multigroup transport theory with $\mathscr{L}$th order anisotropic scattering is discussed. The developed theory reduces the calculation of the reflected and transmitted angular fluxes to a sequence of one-group problems involving only angular fluxes at the boundaries. The theory is then extended to yield similar results at any location within a finite slab. The F_N method is used to establish particularly accurate numerical results for a test problem.

1. INTRODUCTION

In two recent works [1, 2] concerning multigroup transport theory a solution for the case of isotropic scattering and a triangular transfer matrix was developed, and numerical results were reported. Here we extend our previous analysis to include the important effects of anisotropic scattering. We thus consider, for $i = 1, 2, \ldots, M$,

$$\mu \frac{\partial}{\partial z} \psi_i(z, \mu) + \sigma_i \psi_i(z, \mu) = \frac{1}{2} \sum_{j=1}^{i} \sum_{l=0}^{\mathscr{L}} \sigma_{ij}(l) P_l(\mu) \phi_{j,l}(z), \tag{1}$$

where σ_i is the total cross section for group i and $\sigma_{ij}(l) = \sigma_{ij} \beta_{ij}(l)$, with $\beta_{ij}(0) = 1$, denote coefficients in Legendre expansions of the transfer cross sections. In addition $\psi_i(z, \mu)$ represents the angular flux in the ith group and

$$\phi_{j,l}(z) = \int_{-1}^{1} \psi_j(z, \mu) P_l(\mu) \, d\mu. \tag{2}$$

We are concerned here with nonmultiplying, $\sigma_{ii} < \sigma_i$, finite slabs, $z \in [L, R]$, and thus we seek solutions to Eq. (1) subject to the boundary conditions

$$\psi_i(L, \mu) = L_i(\mu), \qquad \mu > 0, \tag{3a}$$

and

$$\psi_i(R, -\mu) = R_i(\mu), \qquad \mu > 0, \tag{3b}$$

where $L_i(\mu)$ and $R_i(\mu)$ are considered specified.

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2. Analysis

In [1] full-range orthogonality properties of appropriate elementary solutions and Green's functions were used to deduce a system of singular-integral equations and constraints for the boundary fluxes. Here we shall develop the equivalent expressions, generalized to include the effects of anisotropic scattering, in a more direct manner. We first change μ to $-\mu$ in Eq. (1), multiply the resulting equation by $\exp(-\sigma_i z/s)$ and integrate over all z to obtain

$$s\mu B_i(\mu, s) - \sigma_i(\mu - s)\int_L^R \psi_i(z, -\mu)\exp(-\sigma_i z/s)\,dz$$
$$= \frac{s}{2}\sum_{j=1}^{i}\sum_{l=0}^{\mathscr{L}} (-1)^l \sigma_{ij}(l)\, P_l(\mu)\, \Phi_{j,l}(s/\sigma_i), \tag{4}$$

where

$$B_i(\mu, s) = \psi_i(L, -\mu)\exp(-\sigma_i L/s) - \psi_i(R, -\mu)\exp(-\sigma_i R/s) \tag{5}$$

and

$$\Phi_{j,l}(s/\sigma_i) = \int_L^R \phi_{j,l}(z)\exp(-\sigma_i z/s)\,dz. \tag{6}$$

We can now multiply Eq. (4) by $(\mu - s)^{-1} P_n(\mu)$, $s \notin [-1, 1]$, and integrate over all μ to find

$$(-1)^n \sigma_i \Phi_{i,n}(s/\sigma_i) + \frac{s}{2}\sum_{j=1}^{i}\sum_{l=0}^{\mathscr{L}} (-1)^l \sigma_{ij}(l)\, \Phi_{j,l}(s/\sigma_i)\int_{-1}^{1} P_n(\mu)\, P_l(\mu)\frac{d\mu}{\mu - s}$$
$$= s\int_{-1}^{1} \mu P_n(\mu)\, B_i(\mu, s)\frac{d\mu}{\mu - s}. \tag{7}$$

We let $g_{i,n}(\xi)$ denote for the ith group the polynomials introduced for one-group theory by Chandrasekhar [3], i.e.,

$$h_{i,n}\xi g_{i,n}(\xi) = (n+1)\, g_{i,n+1}(\xi) + n g_{i,n-1}(\xi) \tag{8}$$

with $g_{i,0}(\xi) = 1$ and

$$h_{i,n} = 2n + 1 - c_i \beta_{ii}(n), \tag{9}$$

where $c_i = \sigma_{ii}/\sigma_i$. On multiplying Eq. (7) by $\beta_{ii}(n)\, g_{i,n}(s)$ and summing over n from 0 to $\mathscr{L}$, we find

$$\sigma_i \sum_{l=0}^{\mathscr{L}} (-1)^l \beta_{ii}(l)\, \Phi_{i,l}(s/\sigma_i)\, F_{i,l}(s) + \frac{1}{c_i}\sum_{j=1}^{i-1}\sum_{l=0}^{\mathscr{L}} (-1)^l \sigma_{ij}(l)\, \Phi_{j,l}(s/\sigma_i)$$
$$\times [F_{i,l}(s) - g_{i,l}(s)] = s\int_{-1}^{1} \mu G_i(s, \mu)\, B_i(\mu, s)\frac{d\mu}{\mu - s}, \tag{10}$$

where

$$F_{i,l}(s) = g_{i,l}(s) + \frac{s}{2} c_i \int_{-1}^{1} G_i(s, \mu) P_l(\mu) \frac{d\mu}{\mu - s} \quad (11)$$

and

$$G_i(s, \mu) = \sum_{l=0}^{\mathscr{L}} \beta_{ii}(l)\, g_{i,l}(s) P_l(\mu). \quad (12)$$

It is not difficult to show that

$$F_{i,l}(s) = \Lambda_i(s) P_l(s), \quad (13)$$

where

$$\Lambda_i(s) = 1 + s \int_{-1}^{1} \psi_i(\mu) \frac{d\mu}{\mu - s}, \quad (14)$$

with

$$\psi_i(\mu) = \tfrac{1}{2} c_i G_i(\mu, \mu), \quad (15)$$

is the one-group dispersion function [4]. We find we can write Eq. (10) as

$$\Lambda_i(s) X_{ii}(s) = \frac{s}{\sigma_i} \int_{-1}^{1} \mu G_i(s, \mu) B_i(\mu, s) \frac{d\mu}{\mu - s} + \frac{1}{\sigma_i} \sum_{j=1}^{i-1} \sigma_{ij} [Y_{ij}(s) - \Delta_i(s) X_{ij}(s)], \quad (16)$$

where

$$X_{ij}(s) = \sum_{l=0}^{\mathscr{L}} (-1)^l \beta_{ij}(l)\, \Phi_{j,l}(s/\sigma_i) P_l(s), \quad (17)$$

$$Y_{ij}(s) = \sum_{l=1}^{\mathscr{L}} (-1)^l \beta_{ij}(l)\, \Phi_{j,l}(s/\sigma_i) E_{i,l}(s) \quad (18)$$

and

$$\Delta_i(s) = \frac{s}{2} \int_{-1}^{1} G_i(\mu, \mu) \frac{d\mu}{\mu - s}. \quad (19)$$

Here the polynomials $E_{i,l}(s)$ are defined by

$$E_{i,l}(s) = (1/c_i)[g_{i,l}(s) - P_l(s)], \quad (20)$$

and with $E_{i,0}(s) = 0$ they can be readily computed from

$$(2l + 1) s E_{i,l}(s) = s\beta_{ii}(l)\, g_{i,l}(s) + (l + 1) E_{i,l+1}(s) + l E_{i,l-1}(s). \quad (21)$$

We note that the functions $\Phi_{j,l}(s/\sigma_i)$ can have essential singularities at the origin, but otherwise, they are analytic in the complex s-plane. The functions $X_{ij}(s)$ and $Y_{ij}(s)$ therefore are, with the exception of the origin, also analytic in the complex s-plane. Thus, on investigating Eq. (16) for the first group, $i = 1$, and assuming that $c_1 \neq 0$, we see that

$$\int_{-1}^{1} \mu G_1(\zeta_{1,m}, \mu) B_1(\mu, \zeta_{1,m}) \frac{d\mu}{\mu - \zeta_{1,m}} = 0, \tag{22}$$

where, in general, $\zeta_{i,m}$, $m = 0, 1, 2, \ldots, 2\kappa_i - 1$ are the zeros of $\Lambda_i(s)$. The left- and right-hand sides of Eq. (16) are analytic in the complex s-plane cut from -1 to 1 along the real axis. Thus, on letting s approach the branch cut and using the Plemelj formulas [5], we find that Eq. (16) yields, for $\nu \in [-1, 1]$,

$$\sigma_1[\lambda_1(\nu) \pm \pi i \nu \psi_1(\nu)] X_{11}(\nu) = \nu P \int_{-1}^{1} \mu G_1(\nu, \mu) B_1(\mu, \nu) \frac{d\mu}{\mu - \nu}$$
$$\pm \pi i \nu^2 G_1(\nu, \nu) B_1(\nu, \nu), \tag{23}$$

where, in general,

$$\lambda_i(\nu) = 1 + \nu P \int_{-1}^{1} \psi_i(\mu) \frac{d\mu}{\mu - \nu}. \tag{24}$$

Thus, for $\nu \in [-1, 1]$, it follows that

$$\sigma_1 c_1 X_{11}(\nu) = 2\nu B_1(\nu, \nu) \tag{25}$$

and

$$\lambda_1(\nu) \nu B_1(\nu, \nu) - \tfrac{1}{2} c_1 \nu P \int_{-1}^{1} \mu G_1(\nu, \mu) B_1(\mu, \nu) \frac{d\mu}{\mu - \nu} = 0. \tag{26}$$

Equations (22) and (26) can be seen to be the system of singular-integral equations and constraints [6, 7] that define the exit fluxes for the first group $\psi_1(L, -\mu)$ and $\psi_1(R, \mu)$, $\mu > 0$, in terms of the incident distributions $L_1(\mu)$ and $R_1(\mu)$. Thus, Eqs. (22) and (26) can be solved numerically or, e.g., by the F_N method [7–9] to establish $B_1(\mu, s)$. In the event that $c_1 = 0$, Eq. (25) yields $B_1(\mu, \mu) = 0$, $|\mu| \in (0, 1]$.

Considering now that $B_1(\mu, s)$ is known, we note that Eq. (16) yields, for $i = 2$,

$$\Lambda_2(s) X_{22}(s) = \frac{s}{\sigma_2} \int_{-1}^{1} \mu G_2(s, \mu) B_2(\mu, s) \frac{d\mu}{\mu - s}$$
$$+ \frac{1}{\sigma_2} \sigma_{21}[Y_{21}(s) - \Lambda_2(s) X_{21}(s)] \tag{27}$$

or, for $\nu \in [-1, 1]$,

$$\lambda_2(\nu)\,\nu B_2(\nu, \nu) - \tfrac{1}{2}c_2 \nu P \int_{-1}^{1} \mu G_2(\nu, \mu)\, B_2(\mu, \nu) \frac{d\mu}{\mu - \nu}$$
$$= \tfrac{1}{2}\sigma_{21}[c_2 Y_{21}(\nu) + X_{21}(\nu)] \tag{28}$$

and

$$\sigma_2 c_2 X_{22}(\nu) = 2\nu B_2(\nu, \nu) - \sigma_{21} X_{21}(\nu). \tag{29}$$

For $c_2 \neq 0$, Eq. (27) yields

$$c_2 \zeta_{2,m} \int_{-1}^{1} \mu G_2(\zeta_{2,m}, \mu)\, B_2(\mu, \zeta_{2,m}) \frac{d\mu}{\mu - \zeta_{2,m}} = -\sigma_{21}[c_2 Y_{21}(\zeta_{2,m}) + X_{21}(\zeta_{2,m})]. \tag{30}$$

For $c_2 = 0$, Eq. (29) yields, for $|\mu| \in (0, 1]$,

$$B_2(\mu, \mu) = (2\mu)^{-1}\, \sigma_{21} X_{21}(\mu), \qquad c_2 = 0, \tag{31}$$

whereas for $c_2 \neq 0$ we can solve Eqs. (28) and (30) to find $B_2(\mu, s)$; of course, for either case we must first compute

$$W_{21}(s) = c_2 Y_{21}(s) + X_{21}(s) \tag{32}$$

which can also be written, if we use Eqs. (17) and (18), as

$$W_{21}(s) = \sum_{l=0}^{\mathscr{L}} (-1)^l \beta_{21}(l)\, \Phi_{1,l}(s/\sigma_2)\, g_{2,l}(s). \tag{33}$$

If we now multiply Eq. (4) by $P_n(\mu)$ and integrate over all μ there results, for $i = 1$,

$$sh_{1,n} \Phi_{1,n}(s/\sigma_1) + (n + 1)\, \Phi_{1,n+1}(s/\sigma_1) + n\Phi_{1,n-1}(s/\sigma_1)$$
$$= -(-1)^n (2n + 1) \frac{s}{\sigma_1} \int_{-1}^{1} \mu P_n(\mu)\, B_1(\mu, s)\, d\mu, \tag{34}$$

which yields

$$\Phi_{1,n}(s/\sigma_1) = (-1)^n g_{1,n}(s)\, \Phi_{1,0}(s/\sigma_1) - (-1)^n D_{1,n}(s), \tag{35}$$

where $D_{1,0}(s) = 0$ and

$$sh_{1,n} D_{1,n}(s) = (n + 1)\, D_{1,n+1}(s) + nD_{1,n-1}(s)$$
$$+ (2n + 1) \frac{s}{\sigma_1} \int_{-1}^{1} \mu P_n(\mu)\, B_1(\mu, s)\, d\mu. \tag{36}$$

Substituting Eq. (35) into Eq. (17), with $i=j=1$, we find

$$\Phi_{1,0}(s/\sigma_1)=G_1^{-1}(s,s)\left[X_{11}(s)+\sum_{l=1}^{\mathscr{L}}\beta_{11}(l)\,D_{1,l}(s)\,P_l(s)\right]. \tag{37}$$

An expression alternative to Eq. (37) that allows the calculation of $\Phi_{1,0}(s/\sigma_1)$ in the event that $G_1(s,s)=0$ is provided in Appendix A. We see from Eqs. (25) and (26) that, for $\nu\in[-1,1]$,

$$X_{11}(\nu)=\frac{1}{\sigma_1}\left\{2\nu B_1(\nu,\nu)+\nu\int_{-1}^{1}\mu G_1(\nu,\mu)[B_1(\mu,\nu)-B_1(\nu,\nu)]\frac{d\mu}{\mu-\nu}\right\} \tag{38}$$

and from Eq. (16) that, for $s\notin[-1,1]$,

$$X_{11}(s)=\frac{s}{\sigma_1}\Lambda_1^{-1}(s)\int_{-1}^{1}\mu G_1(s,\mu)\,B_1(\mu,s)\frac{d\mu}{\mu-s}. \tag{39}$$

It is apparent from Eqs. (22) and (39) that a limiting procedure must be used if $X_{11}(\zeta_{1,m})$ is required. For the case $c_1=0$ we note, since $B_1(\nu,\nu)=0$, $|\nu|\in(0,1]$, that Eqs. (38) and (39) reduce to the following equation for all s:

$$X_{11}(s)=\frac{s}{\sigma_1}\int_{-1}^{1}\mu G_1(s,\mu)\,B_1(\mu,s)\frac{d\mu}{\mu-s},\qquad c_1=0. \tag{40}$$

Finally, we use Eqs. (33) and (35) to conclude that

$$W_{21}(s)=\Phi_{1,0}(s/\sigma_2)\sum_{l=0}^{\mathscr{L}}\beta_{21}(l)\,g_{1,l}(s_{21}s)\,g_{2,l}(s)-\sum_{l=1}^{\mathscr{L}}\beta_{21}(l)\,D_{1,l}(s_{21}s)\,g_{2,l}(s), \tag{41}$$

where $\Phi_{1,0}(s/\sigma_2)$ is available from Eq. (37) and, in general, $s_{ij}=\sigma_j/\sigma_i$. Equations (28) and (30) can now be solved to yield the exit distributions for the second group $\psi_2(L,-\mu)$ and $\psi_2(R,\mu)$, $\mu>0$, in terms of the incident distributions $L_2(\mu)$ and $R_2(\mu)$ and the previously established $B_1(\mu,s)$. Note that in this way we are able to deduce the exit fluxes for the second group directly from the incident distributions for that group and the *boundary* fluxes of the first group.

We now wish to generalize the foregoing and consider the ith group. We assume that the $B_j(\mu,s)$, $j=1,2,\ldots,i-1$, have been established, and we deduce from Eq. (16) that, for $\nu\in[-1,1]$,

$$\lambda_i(\nu)\,\nu B_i(\nu,\nu)-\tfrac{1}{2}c_i\nu P\int_{-1}^{1}\mu G_i(\nu,\mu)\,B_i(\mu,\nu)\frac{d\mu}{\mu-\nu}=\tfrac{1}{2}\sum_{j=1}^{i-1}\sigma_{ij}W_{ij}(\nu) \tag{42}$$

and

$$\sigma_i c_i X_{ii}(\nu)=2\nu B_i(\nu,\nu)-\sum_{j=1}^{i-1}\sigma_{ij}X_{ij}(\nu) \tag{43}$$

and, for $c_i \neq 0$,

$$c_i \zeta_{i,m} \int_{-1}^{1} \mu G_i(\zeta_{i,m}, \mu) B_i(\mu, \zeta_{i,m}) \frac{d\mu}{\mu - \zeta_{i,m}} = -\sum_{j=1}^{i-1} \sigma_{ij} W_{ij}(\zeta_{i,m}), \tag{44}$$

where

$$W_{ij}(s) = \sum_{l=0}^{\mathscr{L}} (-1)^l \beta_{ij}(l) \Phi_{j,l}(s/\sigma_i) g_{i,l}(s). \tag{45}$$

For the special case $c_i = 0$, we see from Eq. (43) that, for $|\mu| \in (0, 1]$,

$$B_i(\mu, \mu) = \frac{1}{2\mu} \sum_{j=1}^{i-1} \sigma_{ij} X_{ij}(\mu), \qquad c_i = 0, \tag{46}$$

whereas for $c_i \neq 0$, the functions $W_{ij}(s)$ rather than just $X_{ij}(\mu)$, $|\mu| \in (0, 1]$, clearly are required before we can solve Eqs. (42) and (44) to find $B_i(\mu, s)$. In analogy with Eq. (34) we now find, for $j = 1, 2, \ldots, i-1$,

$$sh_{j,n} \Phi_{j,n}(s/\sigma_j) + (n+1) \Phi_{j,n+1}(s/\sigma_j) + n\Phi_{j,n-1}(s/\sigma_j) = -(-1)^n M_{j,n}(s), \tag{47}$$

where

$$M_{j,n}(s) = (2n+1) \frac{s}{\sigma_j} \int_{-1}^{1} \mu P_n(\mu) B_j(\mu, s)\, d\mu - (-1)^n \frac{s}{\sigma_j} \sum_{k=1}^{j-1} \sigma_{jk}(n) \Phi_{k,n}(s/\sigma_j). \tag{48}$$

It is clear that we can write

$$\Phi_{j,n}(s/\sigma_j) = (-1)^n g_{j,n}(s) \Phi_{j,0}(s/\sigma_j) - (-1)^n D_{j,n}(s), \tag{49}$$

where $D_{j,0}(s) = 0$ and

$$sh_{j,n} D_{j,n}(s) = (n+1) D_{j,n+1}(s) + nD_{j,n-1}(s) + M_{j,n}(s). \tag{50}$$

On substituting Eq. (49) into Eq. (17), with $i = j$, we find

$$\Phi_{j,0}(s/\sigma_j) = G_j^{-1}(s, s) \left[X_{jj}(s) + \sum_{l=1}^{\mathscr{L}} \beta_{jj}(l) D_{j,l}(s) P_l(s) \right]. \tag{51}$$

Further, we can deduce from Eq. (16) that for $\nu \in [-1, 1]$

$$X_{jj}(\nu) = \frac{1}{\sigma_j} \left\{ 2\nu B_j(\nu, \nu) + \nu \int_{-1}^{1} \mu G_j(\nu, \mu) [B_j(\mu, \nu) - B_j(\nu, \nu)] \frac{d\mu}{\mu - \nu} + \sum_{k=1}^{j-1} \sigma_{jk} Y_{jk}(\nu) \right\} \tag{52}$$

and for $s \notin [-1, 1]$

$$X_{jj}(s) = \frac{1}{\sigma_j} \Lambda_j^{-1}(s) \left\{ s \int_{-1}^{1} \mu G_j(s, \mu) B_j(\mu, s) \frac{d\mu}{\mu - s} + \sum_{k=1}^{j-1} \sigma_{jk} [Y_{jk}(s) - \Lambda_j(s) X_{jk}(s)] \right\}. \tag{53}$$

Again it is apparent that a limiting procedure must be used in the event that $X_{jj}(\zeta_{j,m})$ is required. Finally, if we use Eq. (49) in Eq. (45) we conclude that

$$W_{ij}(s) = \Phi_{j,0}(s/\sigma_i) \sum_{l=0}^{\mathscr{L}} \beta_{ij}(l)\, g_{j,l}(s_{ij} s)\, g_{i,l}(s) - \sum_{l=1}^{\mathscr{L}} \beta_{ij}(l)\, D_{j,l}(s_{ij} s)\, g_{i,l}(s), \tag{54}$$

where $\Phi_{j,0}(s/\sigma_i)$ is available from Eq. (51). We recall that $W_{ij}(\mu) = X_{ij}(\mu)$ for $c_i = 0$, and thus, for this case, Eq. (54) can be used in Eq. (46) to establish the desired result.

3. The F_N Solution

Rather than pursue exact analysis to solve the developed singular-integral equations and constraints for the exit distributions $\psi_i(L, -\mu)$ and $\psi_i(R, \mu)$, $\mu > 0$, we prefer to use the F_N method [7–9] to construct a concise approximate solution. We thus let $\Delta = R - L$, $\Delta_i = \sigma_i \Delta$, and write, for the ith group and $\mu > 0$,

$$\psi_i(L, -\mu) = R_i(\mu) \exp(-\Delta_i/\mu) + \sum_{\alpha=0}^{N} a_{i,\alpha} P_\alpha(2\mu - 1) \tag{55a}$$

and

$$\psi_i(R, \mu) = L_i(\mu) \exp(-\Delta_i/\mu) + \sum_{\alpha=0}^{N} b_{i,\alpha} P_\alpha(2\mu - 1). \tag{55b}$$

If we now use Eqs. (3) and (55) in Eq. (5) we can deduce from Eqs. (42) and (44) that

$$\sum_{\alpha=0}^{N} [a_{i,\alpha} B_{i,\alpha}(\xi) + c_i \exp(-\Delta_i/\xi)\, b_{i,\alpha} A_{i,\alpha}(\xi)] = c_i I_i(\xi) + \sum_{j=1}^{i-1} \sigma_{ij} I_{ij}(\xi) \tag{56a}$$

and

$$\sum_{\alpha=0}^{N} [b_{i,\alpha} B_{i,\alpha}(\xi) + c_i \exp(-\Delta_i/\xi)\, a_{i,\alpha} A_{i,\alpha}(\xi)] = c_i J_i(\xi) + \sum_{j=1}^{i-1} \sigma_{ij} J_{ij}(\xi) \tag{56b}$$

for all $\xi \in P_i = \{\nu_{i,m}\} \cup [0, 1]$. Here $\nu_{i,m}$, $m = 0, 1, 2, \ldots, \kappa_i - 1$, denote the positive discrete eigenvalues relevant to group i,

$$I_i(\xi) = \int_0^1 \mu[L_i(\mu)\, G_i(-\xi, \mu)\, S_i(\Delta, \mu, \xi) + R_i(\mu)\, G_i(\xi, \mu)\, C_i(\Delta, \mu, \xi)]\, d\mu \tag{57a}$$

and

$$J_i(\xi) = \int_0^1 \mu[L_i(\mu)\, G_i(\xi, \mu)\, C_i(\Delta, \mu, \xi) + R_i(\mu)\, G_i(-\xi, \mu)\, S_i(\Delta, \mu, \xi)]\, d\mu, \tag{57b}$$

where

$$S_i(\Delta, \mu, \xi) = (1 - \exp(-\sigma_i \Delta/\mu) \exp(-\sigma_i \Delta/\xi))/(\mu + \xi) \tag{58a}$$

and

$$C_i(\Delta, \mu, \xi) = (\exp(-\sigma_i \Delta/\mu) - \exp(-\sigma_i \Delta/\xi))/(\mu - \xi). \tag{58b}$$

We have also introduced

$$\xi I_{ij}(\xi) = e^{\sigma_i L/\xi} W_{ij}(\xi) \tag{59a}$$

and

$$\xi J_{ij}(\xi) = e^{-\sigma_i R/\xi} W_{ij}(-\xi). \tag{59b}$$

Finally, the functions $A_{i,\alpha}(\xi)$ and $B_{i,\alpha}(\xi)$ required in Eqs. (56) are defined as

$$A_{i,\alpha}(\xi) = \int_0^1 \mu P_\alpha(2\mu - 1)\, G_i(-\xi, \mu) \frac{d\mu}{\mu + \xi}, \qquad \xi \notin [-1, 0), \tag{60}$$

$$B_{i,\alpha}(\xi) = -c_i \int_0^1 \mu P_\alpha(2\mu - 1)\, G_i(\xi, \mu) \frac{d\mu}{\mu - \xi}, \qquad \xi \in \{\nu_{i,m}\}, \tag{61a}$$

and

$$B_{i,\alpha}(\xi) = 2\lambda_i(\xi)\, P_\alpha(2\xi - 1) - c_i P \int_0^1 \mu P_\alpha(2\mu - 1)\, G_i(\xi, \mu) \frac{d\mu}{\mu - \xi}, \qquad \xi \in [0, 1]. \tag{61b}$$

In Appendix B we report some recursion relations that establish an accurate and convenient method for evaluating these basic functions $A_{i,\alpha}(\xi)$ and $B_{i,\alpha}(\xi)$.

Considering the right-hand sides of Eqs. (56), we note from Eqs. (57) that the functions $I_i(\xi)$ and $J_i(\xi)$ are immediately available directly from the boundary data for the ith group. The additional terms $I_{ij}(\xi)$ and $J_{ij}(\xi)$ clearly represent down-scattering contributions to the ith group. We assume now that the constants $\{a_{j,\alpha}\}$

and $\{b_{j,\alpha}\}$, $j=1,2,\ldots,i-1$, have been found so that the approximate results, for $\mu>0$,

$$B_j(\mu,s)=\exp(-\sigma_j L/s)\left\{R_j(\mu)[\exp(-\Delta_j/\mu)-\exp(-\Delta_j/s)]\right.$$

$$\left.+\sum_{\alpha=0}^{N} a_{j,\alpha}P_\alpha(2\mu-1)\right\} \tag{62a}$$

and

$$B_j(-\mu,s)=\exp(-\sigma_j L/s)\left\{L_j(\mu)[1-\exp(-\Delta_j/\mu)\exp(-\Delta_j/s)]-\exp(-\Delta_j/s)\right.$$

$$\left.\times\sum_{\alpha=0}^{N} b_{j,\alpha}P_\alpha(2\mu-1)\right\} \tag{62b}$$

are available for $j=1,2,\ldots,i-1$. Thus, on considering Eqs. (56) at $N+1$ values of $\xi\in P_i$, say $\xi_{i,\beta}$, we can solve the system of linear algebraic equations

$$\sum_{\alpha=0}^{N}\left[a_{i,\alpha}B_{i,\alpha}(\xi_{i,\beta})+c_i\exp(-\Delta_i/\xi_{i,\beta})\,b_{i,\alpha}A_{i,\alpha}(\xi_{i,\beta})\right]$$

$$=c_iI_i(\xi_{i,\beta})+\sum_{j=1}^{i-1}\sigma_{ij}I_{ij}(\xi_{i,\beta}) \tag{63a}$$

and

$$\sum_{\alpha=0}^{N}\left[b_{i,\alpha}B_{i,\alpha}(\xi_{i,\beta})+c_i\exp(-\Delta_i/\xi_{i,\beta})\,a_{i,\alpha}A_{i,\alpha}(\xi_{i,\beta})\right]$$

$$=c_iJ_i(\xi_{i,\beta})+\sum_{j=1}^{i-1}\sigma_{ij}J_{ij}(\xi_{i,\beta}), \tag{63b}$$

for $\beta=0,1,2,\ldots,N$, to find the desired constants for the ith group $\{a_{i,\alpha}\}$ and $\{b_{i,\alpha}\}$ provided we first express $I_{ij}(\xi)$ and $J_{ij}(\xi)$ in terms of known quantities. We therefore proceed to use Eqs. (59) and the various results developed in Section 2 in order to deduce expressions that can be used in a convenient manner to compute the desired $I_{ij}(\xi)$ and $J_{ij}(\xi)$. To be specific we note that for $i=1$ the right-hand sides of Eqs. (63) are known since $I_1(\xi)$ and $J_1(\xi)$ are given by Eqs. (57). Thus, we can solve the system of linear algebraic equations to find $\{a_{1,\alpha}\}$ and $\{b_{1,\alpha}\}$. Considering Eqs. (63) for $i\geqslant 2$, we see that we must compute $I_{ij}(\xi)$ and $J_{ij}(\xi)$ for $j=1,2,\ldots,i-1$, along with $I_i(\xi)$ and $J_i(\xi)$ as given by Eqs. (57), before we can solve the linear system to find $\{a_{i,\alpha}\}$ and $\{b_{i,\alpha}\}$. We find

$$I_{ij}(\xi)=\sum_{l=0}^{\mathscr{L}}(-1)^l\beta_{ij}(l)\,\Phi_{j,l}^{+}(\xi/\sigma_i)\,g_{i,l}(\xi), \tag{64}$$

where

$$\Phi_{j,l}^{+}(\xi/\sigma_i) = (-1)^l g_{j,l}(s_{ij}\xi)\, \Phi_{j,0}^{+}(\xi/\sigma_i) - (-1)^l D_{j,l}^{+}(s_{ij}\xi) \tag{65}$$

with

$$\Phi_{j,0}^{+}(\xi/\sigma_i) = G_j^{-1}(s_{ij}\xi, s_{ij}\xi) \left[X_{jj}^{+}(s_{ij}\xi) + \sum_{l=1}^{\mathscr{L}} \beta_{jj}(l)\, D_{j,l}^{+}(s_{ij}\xi)\, P_l(s_{ij}\xi) \right]. \tag{66}$$

Here $D_{j,l}^{+}(s_{ij}\xi)$, with $D_{j,0}^{+}(s_{ij}\xi) = 0$, are available from

$$s_{ij}\xi h_{j,l} D_{j,l}^{+}(s_{ij}\xi) = (l+1)\, D_{j,l+1}^{+}(s_{ij}\xi) + l D_{j,l-1}^{+}(s_{ij}\xi) + M_{j,l}^{+}(s_{ij}\xi), \tag{67}$$

where

$$M_{j,l}^{+}(s_{ij}\xi) = \frac{2l+1}{\sigma_i} \left\{ K_{j,l}(s_{ij}\xi) + \sum_{\alpha=0}^{l+1} [a_{j,\alpha} + (-1)^l \exp(-\Delta_i/\xi)\, b_{j,\alpha}]\, T_{\alpha,l} \right\}$$
$$- \frac{\xi}{\sigma_i} U_{j,l}(\xi/\sigma_i). \tag{68}$$

In Eq. (68) we have used the definitions

$$K_{j,l}(\xi) = \int_0^1 \mu P_l(\mu) \{ R_j(\mu) [\exp(-\Delta_j/\mu) - \exp(-\Delta_j/\xi)]$$
$$- (-1)^l L_j(\mu) [1 - \exp(-\Delta_j/\mu) \exp(-\Delta_j/\xi)] \}\, d\mu, \tag{69}$$

$$T_{\alpha,l} = \int_0^1 \mu P_\alpha(2\mu - 1)\, P_l(\mu)\, d\mu, \tag{70}$$

and

$$U_{j,l}(\xi/\sigma_i) = (-1)^l \sum_{k=1}^{j-1} \sigma_{jk}(l)\, \Phi_{k,l}^{+}(\xi/\sigma_i). \tag{71}$$

We note that Devaux *et al.* [10] have reported a recursion relation that provides an efficient way to compute the numbers $T_{\alpha,l}$ (see Appendix B). We also point out that $U_{j,l}(\xi/\sigma_i)$ is considered known since all $\Phi_{k,l}^{+}(\xi/\sigma_i)$ required in Eq. (71) have, by necessity, been computed in previous steps. Finally for $s_{ij}\xi \in [0, 1]$, $X_{jj}^{+}(s_{ij}\xi)$ is given by

$$X_{jj}^{+}(s_{ij}\xi) = \frac{1}{\sigma_i} \left\{ I_j(s_{ij}\xi) + [2 - A_{j,0}(s_{ij}\xi)] \sum_{\alpha=0}^{N} a_{j,\alpha} P_\alpha(2 s_{ij}\xi - 1) \right.$$
$$+ \sum_{\alpha=0}^{N} [a_{j,\alpha} G_{j,\alpha}(s_{ij}\xi) - \exp(-\Delta_i/\xi)\, b_{j,\alpha} A_{j,\alpha}(s_{ij}\xi)]$$
$$\left. + s_{ji} \sum_{l=1}^{\mathscr{L}} U_{j,l}(\xi/\sigma_i)\, E_{j,l}(s_{ij}\xi) \right\}, \tag{72}$$

where

$$G_{j,\alpha}(\xi) = \int_0^1 \mu G_j(\xi, \mu) \left[\frac{P_\alpha(2\mu - 1) - P_\alpha(2\xi - 1)}{\mu - \xi}\right] d\mu \tag{73}$$

can be computed effectively from a recursion relation (see Appendix B). For $s_{ij}\xi \notin [0, 1]$ we find

$$X_{jj}^{+}(s_{ij}\xi) = \frac{1}{\sigma_i} \Lambda_j^{-1}(s_{ij}\xi) \left\{ Y_j(s_{ij}\xi) - s_{ji}\Delta_j(s_{ij}\xi) \sum_{l=0}^{\mathscr{L}} U_{j,l}(\xi/\sigma_i)\, P_l(s_{ij}\xi) \right.$$
$$\left. + s_{ji} \sum_{l=1}^{\mathscr{L}} U_{j,l}(\xi/\sigma_i)\, E_{j,l}(s_{ij}\xi) \right\}, \tag{74}$$

where

$$Y_j(\xi) = I_j(\xi) + \sum_{\alpha=0}^{N} [a_{j,\alpha} A_{j,\alpha}(-\xi) - \exp(-\Delta_j/\xi)\, b_{j,\alpha} A_{j,\alpha}(\xi)]. \tag{75}$$

In a similar way we find

$$J_{ij}(\xi) = \sum_{l=0}^{\mathscr{L}} \beta_{ij}(l)\, \Phi_{j,l}^{-}(\xi/\sigma_i)\, g_{i,l}(\xi), \tag{76}$$

where

$$\Phi_{j,l}^{-}(\xi/\sigma_i) = g_{j,l}(s_{ij}\xi)\, \Phi_{j,0}^{-}(\xi/\sigma_i) - (-1)^l D_{j,l}^{-}(s_{ij}\xi) \tag{77}$$

with

$$\Phi_{j,0}^{-}(\xi/\sigma_i) = G_j^{-1}(s_{ij}\xi, s_{ij}\xi) \left[X_{jj}^{-}(s_{ij}\xi) + \sum_{l=1}^{\mathscr{L}} (-1)^l \beta_{jj}(l)\, D_{j,l}^{-}(s_{ij}\xi)\, P_l(s_{ij}\xi) \right]. \tag{78}$$

Here $D_{j,l}^{-}(s_{ij}\xi)$, with $D_{j,0}^{-}(s_{ij}\xi) = 0$, are available from

$$-s_{ij}\xi h_{j,l} D_{j,l}^{-}(s_{ij}\xi) = (l+1)\, D_{j,l+1}^{-}(s_{ij}\xi) + l D_{j,l-1}^{-}(s_{ij}\xi) + M_{j,l}^{-}(s_{ij}\xi), \tag{79}$$

where

$$M_{j,l}^{-}(s_{ij}\xi) = \frac{2l+1}{\sigma_i} \left\{ N_{j,l}(s_{ij}\xi) - \sum_{\alpha=0}^{l+1} [(-1)^l b_{j,\alpha} + \exp(-\Delta_i/\xi)\, a_{j,\alpha}]\, T_{\alpha,l} \right\}$$
$$+ (-1)^l (\xi/\sigma_i)\, V_{j,l}(\xi/\sigma_i). \tag{80}$$

In Eq. (80) we have used the definitions

$$N_{j,l}(\xi) = \int_0^1 \mu P_l(\mu) \{ R_j(\mu)[1 - \exp(-\Delta_j/\mu) \exp(-\Delta_j/\xi)]$$
$$- (-1)^l L_j(\mu)[\exp(-\Delta_j/\mu) - \exp(-\Delta_j/\xi)] \}\, d\mu \tag{81}$$

and

$$V_{j,l}(\xi/\sigma_i) = \sum_{k=1}^{j-1} \sigma_{jk}(l)\, \Phi_{k,l}^{-}(\xi/\sigma_i) \tag{82}$$

where the latter, as discussed before, is available from previous steps. Finally, for $s_{ij}\xi \in [0, 1]$, $X_{jj}^{-}(s_{ij}\xi)$ is given by

$$\begin{aligned} X_{jj}^{-}(s_{ij}\xi) = \frac{1}{\sigma_i} \Bigg\{ & J_j(s_{ij}\xi) + [2 - A_{j,0}(s_{ij}\xi)] \sum_{\alpha=0}^{N} b_{j,\alpha} P_\alpha(2s_{ij}\xi - 1) \\ & + \sum_{\alpha=0}^{N} [b_{j,\alpha} G_{j,\alpha}(s_{ij}\xi) - \exp(-\Delta_i/\xi)\, a_{j,\alpha} A_{j,\alpha}(s_{ij}\xi)] \\ & + s_{ji} \sum_{l=1}^{\mathscr{L}} V_{j,l}(\xi/\sigma_i)\, E_{j,l}(s_{ij}\xi) \Bigg\}. \end{aligned} \tag{83}$$

For $s_{ij}\xi \notin [0, 1]$ we find

$$\begin{aligned} X_{jj}^{-}(s_{ij}\xi) = \frac{1}{\sigma_i} \Lambda_j^{-1}(s_{ij}\xi) \Bigg\{ & \Xi_j(s_{ij}\xi) - s_{ji}\Delta_j(s_{ij}\xi) \sum_{l=0}^{\mathscr{L}} V_{j,l}(\xi/\sigma_i)\, P_l(s_{ij}\xi) \\ & + s_{ji} \sum_{l=1}^{\mathscr{L}} V_{j,l}(\xi/\sigma_i)\, E_{j,l}(s_{ij}\xi) \Bigg\}, \end{aligned} \tag{84}$$

where

$$\Xi_j(\xi) = J_j(\xi) + \sum_{\alpha=0}^{N} [b_{j,\alpha} A_{j,\alpha}(-\xi) - \exp(-\Delta_j/\xi)\, a_{j,\alpha} A_{j,\alpha}(\xi)]. \tag{85}$$

Having developed the F_N method to find the surface fluxes, we now demonstrate how a slight modification of the analysis of Section 2 and the F_N method can be used to compute accurately the angular fluxes for all $z \in (L, R)$.

4. The Interior Angular Fluxes

If we change μ to $-\mu$ in Eq. (1), multiply the resulting equation by $\exp(-\sigma_i z/s)$, and integrate over z from z_1 to z_2, with $L \leqslant z_1 < z_2 \leqslant R$, we obtain an equation similar to Eq. (4), viz.

$$\begin{aligned} & s\mu B_i^*(\mu, s) - \sigma_i(\mu - s) \int_{z_1}^{z_2} \psi_i(z, -\mu) \exp(-\sigma_i z/s)\, dz \\ & = \frac{s}{2} \sum_{j=1}^{i} \sum_{l=0}^{\mathscr{L}} (-1)^l \sigma_{ij}(l)\, P_l(\mu)\, \Phi_{j,l}^*(s/\sigma_i) \end{aligned} \tag{86}$$

where

$$B_i^*(\mu, s) = \psi_i(z_1, -\mu)\exp(-\sigma_i z_1/s) - \psi_i(z_2, -\mu)\exp(-\sigma_i z_2/s) \tag{87}$$

and

$$\Phi_{j,l}^*(s/\sigma_i) = \int_{z_1}^{z_2} \phi_{j,l}(z)\exp(-\sigma_i z/s)\,dz. \tag{88}$$

We can follow the development discussed in Section 2 to obtain from Eq. (86) generalizations of Eqs. (42)–(44). Thus, for the ith group and for $\nu \in [-1, 1]$ we find

$$\lambda_i(\nu)\,\nu B_i^*(\nu, \nu) - \tfrac{1}{2}c_i\nu P\int_{-1}^{1} \mu G_i(\nu, \mu)\,B_i^*(\mu, \nu)\frac{d\mu}{\mu - \nu} = \tfrac{1}{2}\sum_{j=1}^{i-1} \sigma_{ij} W_{ij}^*(\nu) \tag{89}$$

and

$$\sigma_i c_i X_{ii}^*(\nu) = 2\nu B_i^*(\nu, \nu) - \sum_{j=1}^{i-1} \sigma_{ij} X_{ij}^*(\nu) \tag{90}$$

and, for $c_i \neq 0$,

$$c_i\zeta_{i,m}\int_{-1}^{1} \mu G_i(\zeta_{i,m}, \mu)\,B_i^*(\mu, \zeta_{i,m})\frac{d\mu}{\mu - \zeta_{i,m}} = -\sum_{j=1}^{i-1} \sigma_{ij} W_{ij}^*(\zeta_{i,m}), \tag{91}$$

where

$$W_{ij}^*(s) = \sum_{l=0}^{\mathscr{L}} (-1)^l \beta_{ij}(l)\,\Phi_{j,l}^*(s/\sigma_i)\,g_{i,l}(s) \tag{92}$$

and

$$X_{ij}^*(s) = \sum_{l=0}^{\mathscr{L}} (-1)^l \beta_{ij}(l)\,\Phi_{j,l}^*(s/\sigma_i)\,P_l(s). \tag{93}$$

For the special case $c_i = 0$, we find, for $|\mu| \in (0, 1]$,

$$B_i^*(\mu, \mu) = \frac{1}{2\mu}\sum_{j=1}^{i-1} \sigma_{ij} X_{ij}^*(\mu). \tag{94}$$

We note that the expressions developed in Section 2 for the computation of $\Phi_{j,l}(s/\sigma_i)$ can be generalized so that $\Phi_{j,l}^*(s/\sigma_i)$ can be found in a similar manner.

We now let $z_1 = z$, $z_2 = R$, and then use Eqs. (3b), (55b), and the approximations, for $\mu > 0$,

$$\psi_i(z, -\mu) = R_i(\mu)\exp[-\sigma_i(R - z)/\mu] + \sum_{\alpha=0}^{N} c_{i,\alpha}(z)\,P_\alpha(2\mu - 1) \tag{95a}$$

and

$$\psi_l(z,\mu) = L_l(\mu)\exp[-\sigma_l(z-L)/\mu] + \sum_{\alpha=0}^{N} d_{l,\alpha}(z) P_\alpha(2\mu - 1) \tag{95b}$$

in Eq. (87) to deduce from Eqs. (89) and (91) the first of our F_N equations. Similarly we can take $z_1 = L$, $z_2 = z$, and then use Eqs. (3a), (55a), and (95) in Eq. (87) to deduce from Eqs. (89) and (91) the second of the F_N equations. Thus, for $z \in (L, R)$ and $\xi \in P_l$ we find

$$\begin{aligned}\sum_{\alpha=0}^{N} &[c_{l,\alpha}(z) B_{l,\alpha}(\xi) - c_l d_{l,\alpha}(z) A_{l,\alpha}(\xi)] \\ &= c_l I_l(z,\xi) + \sum_{j=1}^{l-1} \sigma_{lj} I_{lj}(z,\xi) \\ &\quad - c_l \exp[-\sigma_l(R-z)/\xi] \sum_{\alpha=0}^{N} b_{l,\alpha} A_{l,\alpha}(\xi)\end{aligned} \tag{96a}$$

and

$$\begin{aligned}\sum_{\alpha=0}^{N} &[d_{l,\alpha}(z) B_{l,\alpha}(\xi) - c_l c_{l,\alpha}(z) A_{l,\alpha}(\xi)] \\ &= c_l J_l(z,\xi) + \sum_{j=1}^{l-1} \sigma_{lj} J_{lj}(z,\xi) \\ &\quad - c_l \exp[-\sigma_l(z-L)/\xi] \sum_{\alpha=0}^{N} a_{l,\alpha} A_{l,\alpha}(\xi).\end{aligned} \tag{96b}$$

Here

$$\begin{aligned}I_l(z,\xi) = \int_0^1 \mu[&L_l(\mu) G_l(-\xi,\mu)\exp[-\sigma_l(z-L)/\mu]\, S_l(R-z,\mu,\xi) \\ &+ R_l(\mu) G_l(\xi,\mu) C_l(R-z,\mu,\xi)]\, d\mu,\end{aligned} \tag{97a}$$

$$\begin{aligned}J_l(z,\xi) = \int_0^1 \mu[&L_l(\mu) G_l(\xi,\mu) C_l(z-L,\mu,\xi) \\ &+ R_l(\mu) G_l(-\xi,\mu)\exp[-\sigma_l(R-z)/\mu]\, S_l(z-L,\mu,\xi)]\, d\mu,\end{aligned} \tag{97b}$$

$$\xi I_{lj}(z,\xi) = e^{\sigma_l z/\xi} W_{lj}^*(\xi), \qquad z_1 = z \quad \text{and} \quad z_2 = R, \tag{98a}$$

and

$$\xi J_{lj}(z,\xi) = e^{-\sigma_l z/\xi} W_{lj}^*(-\xi), \qquad z_1 = L \quad \text{and} \quad z_2 = z. \tag{98b}$$

Investigating the right-hand sides of Eqs. (96), we note from Eqs. (97) that $I_l(z,\xi)$ and $J_l(z,\xi)$ are available, for any $z \in (L, R)$, directly from the boundary data for the

ith group. The terms $I_{ij}(z, \xi)$ and $J_{ij}(z, \xi)$ represent, as before, down-scattering contributions to the ith group, and at this point the constants $\{a_{i,\alpha}\}$ and $\{b_{i,\alpha}\}$ have already been established. Thus, on considering Eqs. (96) at the same $N+1$ values of $\xi \in P_i$ used in Eqs. (63), namely $\xi_{i,\beta}$, we can solve the system of linear algebraic equations

$$\sum_{\alpha=0}^{N} [c_{i,\alpha}(z) B_{i,\alpha}(\xi_{i,\beta}) - c_i d_{i,\alpha}(z) A_{i,\alpha}(\xi_{i,\beta})]$$
$$= c_i I_i(z, \xi_{i,\beta}) + \sum_{j=1}^{i-1} \sigma_{ij} I_{ij}(z, \xi_{i,\beta}) - c_i \exp[-\sigma_i(R-z)/\xi_{i,\beta}]$$
$$\times \sum_{\alpha=0}^{N} b_{i,\alpha} A_{i,\alpha}(\xi_{i,\beta}) \tag{99a}$$

and

$$\sum_{\alpha=0}^{N} [d_{i,\alpha}(z) B_{i,\alpha}(\xi_{i,\beta}) - c_i c_{i,\alpha}(z) A_{i,\alpha}(\xi_{i,\beta})]$$
$$= c_i J_i(z, \xi_{i,\beta}) + \sum_{j=1}^{i-1} \sigma_{ij} J_{ij}(z, \xi_{i,\beta}) - c_i \exp[-\sigma_i(z-L)/\xi_{i,\beta}]$$
$$\times \sum_{\alpha=0}^{N} a_{i,\alpha} A_{i,\alpha}(\xi_{i,\beta}) \tag{99b}$$

for $\beta = 0, 1, 2,..., N$ to find $\{c_{i,\alpha}(z)\}$ and $\{d_{i,\alpha}(z)\}$ for selected values of $z \in (L, R)$ provided we first express $I_{ij}(z, \xi)$ and $J_{ij}(z, \xi)$ in terms of known quantities. We observe that the functions $A_{i,\alpha}(\xi)$ and $B_{i,\alpha}(\xi)$ appearing in Eqs. (99) are the same as used in Eqs. (63), and thus, the only new quantities to be evaluated are $I_i(z, \xi)$, $J_i(z, \xi)$, $I_{ij}(z, \xi)$, and $J_{ij}(z, \xi)$. Further we note that the matrix of coefficients in Eqs. (99) is independent of z so that the solutions to these equations for many values of z can be obtained with one matrix inversion. We now summarize the equations that can be used to compute the desired $I_{ij}(z, \xi)$ and $J_{ij}(z, \xi)$. We find

$$I_{ij}(z, \xi) = \sum_{l=0}^{\mathscr{L}} (-1)^l \beta_{ij}(l)\, \Phi_{j,l}^{+}(z, \xi/\sigma_i)\, g_{i,l}(\xi), \tag{100}$$

where

$$\Phi_{j,l}^{+}(z, \xi/\sigma_i) = (-1)^l g_{j,l}(s_{ij}\xi)\, \Phi_{j,0}^{+}(z, \xi/\sigma_i) - (-1)^l D_{j,l}^{+}(z, s_{ij}\xi) \tag{101}$$

with

$$\Phi_{j,0}^{+}(z, \xi/\sigma_i) = G_j^{-1}(s_{ij}\xi, s_{ij}\xi) \left[X_{jj}^{+}(z, s_{ij}\xi) + \sum_{l=1}^{\mathscr{L}} \beta_{jj}(l)\, D_{j,l}^{+}(z, s_{ij}\xi)\, P_l(s_{ij}\xi)\right]. \tag{102}$$

Here $D^+_{j,l}(z, s_{ij}\xi)$, with $D^+_{j,0}(z, s_{ij}\xi)$, are available from

$$s_{ij}\xi h_{j,l} D^+_{j,l}(z, s_{ij}\xi) = (l+1)\, D^+_{j,l+1}(z, s_{ij}\xi) + l D^+_{j,l-1}(z, s_{ij}\xi) + M^+_{j,l}(z, s_{ij}\xi), \tag{103}$$

where

$$M^+_{j,l}(z, s_{ij}\xi) = \frac{2l+1}{\sigma_i} \left\{ K_{j,l}(z, s_{ij}\xi) + \sum_{\alpha=0}^{l+1} [c_{j,\alpha}(z) - (-1)^l d_{j,\alpha}(z) + (-1)^l \exp[-\sigma_i(R-z)/\xi]\, b_{j,\alpha}]\, T_{\alpha,l} \right\} - \frac{\xi}{\sigma_i} U_{j,l}(z, \xi/\sigma_i). \tag{104}$$

In Eq. (104) we have used the definitions

$$K_{j,l}(z, \xi) = \int_0^1 \mu P_l(\mu) \{R_j(\mu)[\exp[-\sigma_j(R-z)/\mu] - \exp[-\sigma_j(R-z)/\xi]] - (-1)^l L_j(\mu) \exp[-\sigma_j(z-L)/\mu] \times [1 - \exp[-\sigma_j(R-z)/\mu] \exp[-\sigma_j(R-z)/\xi]]\}\, d\mu \tag{105}$$

and

$$U_{j,l}(z, \xi/\sigma_i) = (-1)^l \sum_{k=1}^{j-1} \sigma_{jk}(l)\, \Phi^+_{k,l}(z, \xi/\sigma_i). \tag{106}$$

Finally, for $s_{ij}\xi \in [0, 1]$, $X^+_{jj}(z, s_{ij}\xi)$ is given by

$$X^+_{jj}(z, s_{ij}\xi) = \frac{1}{\sigma_i} \left\{ I_j(z, s_{ij}\xi) + [2 - A_{j,0}(s_{ij}\xi)] \sum_{\alpha=0}^{N} c_{j,\alpha}(z)\, P_\alpha(2s_{ij}\xi - 1) + \sum_{\alpha=0}^{N} [c_{j,\alpha}(z)\, G_{j,\alpha}(s_{ij}\xi) + \{d_{j,\alpha}(z) - \exp[-\sigma_i(R-z)/\xi]\, b_{j,\alpha}\} A_{j,\alpha}(s_{ij}\xi)] + s_{ji} \sum_{l=1}^{\mathscr{L}} U_{j,l}(z, \xi/\sigma_i)\, E_{j,l}(s_{ij}\xi) \right\}. \tag{107}$$

For $s_{ij}\xi \notin [0, 1]$ we find

$$X^+_{jj}(z, s_{ij}\xi) = \frac{1}{\sigma_i} \Lambda_j^{-1}(s_{ij}\xi) \left\{ Y_j(z, s_{ij}\xi) - s_{ji} \Delta_j(s_{ij}\xi) \sum_{l=0}^{\mathscr{L}} U_{j,l}(z, \xi/\sigma_i)\, P_l(s_{ij}\xi) + s_{ji} \sum_{l=1}^{\mathscr{L}} U_{j,l}(z, \xi/\sigma_i)\, E_{j,l}(s_{ij}\xi) \right\}, \tag{108}$$

where

$$\Upsilon_j(z, \xi) = I_j(z, \xi) + \sum_{\alpha=0}^{N} [c_{j,\alpha}(z) A_{j,\alpha}(-\xi) + \{d_{j,\alpha}(z) - \exp[-\sigma_j(R - z)/\xi]\, b_{j,\alpha}\} A_{j,\alpha}(\xi)]. \quad (109)$$

In a similar way we find

$$J_{ij}(z, \xi) = \sum_{l=0}^{\mathscr{L}} \beta_{ij}(l)\, \Phi_{j,l}^{-}(z, \xi/\sigma_i)\, g_{i,l}(\xi), \quad (110)$$

where

$$\Phi_{j,l}^{-}(z, \xi/\sigma_i) = g_{j,l}(s_{ij}\xi)\, \Phi_{j,0}^{-}(z, \xi/\sigma_i) - (-1)^l D_{j,l}^{-}(z, s_{ij}\xi) \quad (111)$$

with

$$\Phi_{j,0}^{-}(z, \xi/\sigma_i) = G_j^{-1}(s_{ij}\xi, s_{ij}\xi) \left[X_{jj}^{-}(z, s_{ij}\xi) + \sum_{l=1}^{\mathscr{L}} (-1)^l \beta_{jj}(l)\, D_{j,l}^{-}(z, s_{ij}\xi)\, P_l(s_{ij}\xi) \right]. \quad (112)$$

Here $D_{j,l}^{-}(z, s_{ij}\xi)$, with $D_{j,0}^{-}(z, s_{ij}\xi) = 0$, are available from

$$-s_{ij}\xi h_{j,l} D_{j,l}^{-}(z, s_{ij}\xi) = (l+1)\, D_{j,l+1}^{-}(z, s_{ij}\xi) + l D_{j,l-1}^{-}(z, s_{ij}\xi) + M_{j,l}^{-}(z, s_{ij}\xi), \quad (113)$$

where

$$M_{j,l}^{-}(z, s_{ij}\xi) = \frac{2l+1}{\sigma_i} \left\{ N_{j,l}(z, s_{ij}\xi) + \sum_{\alpha=0}^{l+1} [c_{j,\alpha}(z) - (-1)^l d_{j,\alpha}(z) - \exp[-\sigma_i(z - L)/\xi]\, a_{j,\alpha}]\, T_{\alpha,l} \right\} + (-1)^l \frac{\xi}{\sigma_i} V_{j,l}(z, \xi/\sigma_i). \quad (114)$$

In Eq. (114) we have used the definitions

$$N_{j,l}(z, \xi) = \int_0^1 \mu P_l(\mu) \{R_j(\mu) \exp[-\sigma_j(R - z)/\mu] \times [1 - \exp[-\sigma_j(z - L)/\mu] \exp[-\sigma_j(z - L)/\xi]] - (-1)^l L_j(\mu)[\exp[-\sigma_j(z - L)/\mu] - \exp[-\sigma_j(z - L)/\xi]]\}\, d\mu \quad (115)$$

and

$$V_{j,l}(z, \xi/\sigma_i) = \sum_{k=1}^{j-1} \sigma_{jk}(l)\, \Phi_{k,l}^-(z, \xi/\sigma_i). \tag{116}$$

Finally, for $s_{ij}\xi \in [0, 1]$, $X_{jj}^-(z, s_{ij}\xi)$ is given by

$$\begin{aligned} X_{jj}^-(z, s_{ij}\xi) &= \frac{1}{\sigma_i} \Bigg\{ J_j(z, s_{ij}\xi) + [2 - A_{j,0}(s_{ij}\xi)] \sum_{\alpha=0}^{N} d_{j,\alpha}(z)\, P_\alpha(2s_{ij}\xi - 1) \\ &\quad + \sum_{\alpha=0}^{N} [d_{j,\alpha}(z)\, G_{j,\alpha}(s_{ij}\xi) + \{c_{j,\alpha}(z) - \exp[-\sigma_i(z - L)/\xi]\, a_{j,\alpha}\}\, A_{j,\alpha}(s_{ij}\xi)] \\ &\quad + s_{ji} \sum_{l=1}^{\mathscr{L}} V_{j,l}(z, \xi/\sigma_i)\, E_{j,l}(s_{ij}\xi) \Bigg\}. \end{aligned} \tag{117}$$

For $s_{ij}\xi \notin [0, 1]$ we find

$$\begin{aligned} X_{jj}^-(z, s_{ij}\xi) = \frac{1}{\sigma_i} \Lambda_j^{-1}(s_{ij}\xi) \Bigg\{ & \Xi_j(z, s_{ij}\xi) \\ & - s_{ji}\Delta_j(s_{ij}\xi) \sum_{l=0}^{\mathscr{L}} V_{j,l}(z, \xi/\sigma_i)\, P_l(s_{ij}\xi) \\ & + s_{ji} \sum_{l=1}^{\mathscr{L}} V_{j,l}(z, \xi/\sigma_i)\, E_{j,l}(s_{ij}\xi) \Bigg\}, \end{aligned} \tag{118}$$

where

$$\begin{aligned} \Xi_j(z, \xi) = J_j(z, \xi) + \sum_{\alpha=0}^{N} & [d_{j,\alpha}(z)\, A_{j,\alpha}(-\xi) \\ & + \{c_{j,\alpha}(z) - \exp[-\sigma_j(z - L)/\xi]\, a_{j,\alpha}\}\, A_{j,\alpha}(\xi)]. \end{aligned} \tag{119}$$

5. Numerical Results

In order to demonstrate the computational merit of our solution we now consider a 20-group albedo problem with a 10th-order Legendre expansion of the scattering law. A 20-cm thick slab has an isotropically incident distribution of radiation only in the first group and only on the surface at $z = L$, i.e., for $\mu > 0$

$$L_i(\mu) = \delta_{i,1} \tag{120a}$$

and

$$R_i(\mu) = 0. \tag{120b}$$

To facilitate the data handling we use a fictitious cross-section set (in units of cm^{-1}) defined, for $i = 1, 2,..., 20$, by

$$\sigma_i = (i/10) - 0.15\delta_{i,5} - 0.15\delta_{i,10} \tag{121a}$$

and

$$\sigma_{ij}(l) = (2l+1)j/[100(i-j+1)](g_{ij})^l, \qquad j = 1, 2,..., i \quad \text{and} \quad l = 0, 1,..., 10, \tag{121b}$$

where

$$g_{ij} = 0.7 - (i+j)/200. \tag{121c}$$

The scattering law defined by Eqs. (121b) and (121c) is a truncated version of the Henyey–Greenstein phase function introduced in the field of radiative transfer [11]. The Henyey–Greenstein phase function is characterized by one parameter g which is a measure of the degree of anisotropy, i.e., $g \to 1$ implies forward scattering while $g \to -1$ implies backward scattering. For a monoenergetic problem g corresponds to the average cosine of the scattering angle. In our problem the values of g given by Eq. (121c) correspond to moderate forward scattering and were chosen in order to avoid negative values in the scattering law (with $\mathscr{L} = 10$).

In solving the systems of linear algebraic equations given by Eqs. (63) and (99) we have used, for various orders of the F_N approximation, the collocation scheme

$$\xi_{i,\beta} = \nu_{i,\beta}, \qquad \beta = 0, 1, 2,..., \kappa_i - 1, \tag{122a}$$

and

$$\xi_{i,\beta} = \tfrac{1}{2} + \tfrac{1}{2}\cos\{(2\beta - 2\kappa_i + 1)\pi/[2(N+1-\kappa_i)]\}, \qquad \beta = \kappa_i, \kappa_i + 1,..., N. \tag{122b}$$

The points given by Eq. (122b) are the zeros of the Chebyshev polynomial of the first kind $T_{N+1-\kappa_i}(2x-1)$. Based on the results of our computations which closely followed the technique discussed by Siewert [12] we have concluded that there is only one pair of discrete eigenvalues relevant to each group of the considered problem, and thus we list in Table I the positive eigenvalue for each group. We list

TABLE I

The Positive Discrete Eigenvalue Basic to Each Group

i	$\nu_{i,0}$	i	$\nu_{i,0}$	i	$\nu_{i,0}$	i	$\nu_{i,0}$
1	1.014675230187	6	1.010101983620	11	1.006629121797	16	1.004095374943
2	1.013664030621	7	1.009324801731	12	1.006052161369	17	1.003687842667
3	1.012702645157	8	1.008590208601	13	1.005511523631	18	1.003310515972
4	1.011789497866	9	1.007896906406	14	1.005005991723	19	1.002962148042
5	1.024569285561	10	1.011201112487	15	1.004534348940	20	1.002641480584

our converged results for the exit angular fluxes in Tables II–V. Further converged results for the angular fluxes at various positions inside the slab are shown in Tables VI–XI. We note that to compute the angular fluxes accurately for all μ we used a recently proposed technique [13]. First the functions $B_{i,\alpha}(\xi)$ defined by Eqs. (61) are expressed as

$$B_{i,\alpha}(\xi) = 2P_\alpha(2\xi - 1) - c_i\{[2 - A_{i,0}(\xi)]\, P_\alpha(2\xi - 1) + G_{i,\alpha}(\xi)\} \tag{123}$$

and this relation can be used in Eqs. (56) for $\xi = \mu \in [0, 1]$ to find the following alternative expressions for the emerging fluxes:

$$\psi_i(L, -\mu) = R_i(\mu)\exp(-\Delta_i/\mu) + \frac{c_i}{2}\Big\{ I_i(\mu) + [2 - A_{i,0}(\mu)] \sum_{\alpha=0}^{N} a_{i,\alpha} P_\alpha(2\mu - 1) + \sum_{\alpha=0}^{N} [a_{i,\alpha} G_{i,\alpha}(\mu) - \exp(-\Delta_i/\mu)\, b_{i,\alpha} A_{i,\alpha}(\mu)] \Big\} + \tfrac{1}{2}\sum_{j=1}^{i-1} \sigma_{ij} I_{ij}(\mu) \tag{124a}$$

and

$$\psi_i(R, \mu) = L_i(\mu)\exp(-\Delta_i/\mu) + \frac{c_i}{2}\Big\{ J_i(\mu) + [2 - A_{i,0}(\mu)] \sum_{\alpha=0}^{N} b_{i,\alpha} P_\alpha(2\mu - 1) + \sum_{\alpha=0}^{N} [b_{i,\alpha} G_{i,\alpha}(\mu) - \exp(-\Delta_i/\mu)\, a_{i,\alpha} A_{i,\alpha}(\mu)] \Big\} + \tfrac{1}{2}\sum_{j=1}^{i-1} \sigma_{ij} J_{ij}(\mu). \tag{124b}$$

In a similar way we find for the interior fluxes

$$\psi_i(z, -\mu) = R_i(\mu)\exp[-\sigma_i(R - z)/\mu] + \frac{c_i}{2}\Big\{ I_i(z, \mu) + [2 - A_{i,0}(\mu)] \sum_{\alpha=0}^{N} c_{i,\alpha}(z)\, P_\alpha(2\mu - 1) + \sum_{\alpha=0}^{N} [c_{i,\alpha}(z)\, G_{i,\alpha}(\mu) + \{d_{i,\alpha}(z) - \exp[-\sigma_i(R - z)/\mu]\, b_{i,\alpha}\} A_{i,\alpha}(\mu)] \Big\} + \tfrac{1}{2}\sum_{j=1}^{i-1} \sigma_{ij} I_{ij}(z, \mu) \tag{125a}$$

and

$$\psi_i(z, \mu) = L_i(\mu)\exp[-\sigma_i(z - L)/\mu] + \frac{c_i}{2}\Big\{ J_i(z, \mu) + [2 - A_{i,0}(\mu)] \sum_{\alpha=0}^{N} d_{i,\alpha}(z)\, P_\alpha(2\mu - 1) + \sum_{\alpha=0}^{N} [d_{i,\alpha}(z)\, G_{i,\alpha}(\mu) + \{c_{i,\alpha}(z) - \exp[-\sigma_i(z - L)/\mu]\, a_{i,\alpha}\} A_{i,\alpha}(\mu)] \Big\} + \tfrac{1}{2}\sum_{j=1}^{i-1} \sigma_{ij} J_{ij}(z, \mu). \tag{125b}$$

TABLE II

The Exit Angular Fluxes $\psi_i(L, -\mu)$ for $i = 1$–10

μ	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$	$i = 7$	$i = 8$	$i = 9$	$i = 10$
0	5.0885(−2)	1.2999(−2)	5.8860(−3)	3.3641(−3)	3.1535(−3)	1.5457(−3)	1.1467(−3)	8.8628(−4)	7.0652(−4)	6.8284(−4)
0.1	2.8195(−2)	8.6681(−3)	4.2785(−3)	2.5789(−3)	2.4685(−3)	1.2701(−3)	9.6248(−4)	7.5717(−4)	6.1275(−4)	6.0047(−4)
0.2	1.8568(−2)	6.2371(−3)	3.2310(−3)	2.0122(−3)	1.9565(−3)	1.0397(−3)	7.9986(−4)	6.3758(−4)	5.2196(−4)	5.1654(−4)
0.3	1.3061(−2)	4.6596(−3)	2.4996(−3)	1.5956(−3)	1.5724(−3)	8.5661(−4)	6.6715(−4)	5.3764(−4)	4.4447(−4)	4.4359(−4)
0.4	9.6343(−3)	3.5983(−3)	1.9835(−3)	1.2912(−3)	1.2862(−3)	7.1511(−4)	5.6261(−4)	4.5755(−4)	3.8138(−4)	3.8335(−4)
0.5	7.3536(−3)	2.8516(−3)	1.6080(−3)	1.0640(−3)	1.0690(−3)	6.0504(−4)	4.8015(−4)	3.9355(−4)	3.3036(−4)	3.3407(−4)
0.6	5.7554(−3)	2.3033(−3)	1.3241(−3)	8.8871(−4)	8.9923(−4)	5.1717(−4)	4.1361(−4)	3.4138(−4)	2.8839(−4)	2.9317(−4)
0.7	4.6063(−3)	1.8921(−3)	1.1055(−3)	7.5096(−4)	7.6454(−4)	4.4586(−4)	3.5904(−4)	2.9821(−4)	2.5337(−4)	2.5882(−4)
0.8	3.7653(−3)	1.5814(−3)	9.3667(−4)	6.4282(−4)	6.5749(−4)	3.8823(−4)	3.1451(−4)	2.6267(−4)	2.2431(−4)	2.3009(−4)
0.9	3.1192(−3)	1.3378(−3)	8.0287(−4)	5.5637(−4)	5.7091(−4)	3.4135(−4)	2.7806(−4)	2.3339(−4)	2.0022(−4)	2.0608(−4)
1	2.6287(−3)	1.1453(−3)	6.9424(−4)	4.8472(−4)	4.9900(−4)	3.0128(−4)	2.4661(−4)	2.0793(−4)	1.7914(−4)	1.8504(−4)

TABLE III

The Exit Angular Fluxes $\psi_i(L, -\mu)$ for $i = 11$–20

μ	$i = 11$	$i = 12$	$i = 13$	$i = 14$	$i = 15$	$i = 16$	$i = 17$	$i = 18$	$i = 19$	$i = 20$
0	4.8231(−4)	4.0797(−4)	3.4993(−4)	3.0368(−4)	2.6619(−4)	2.3537(−4)	2.0970(−4)	1.8810(−4)	1.6974(−4)	1.5400(−4)
0.1	4.2927(−4)	3.6660(−4)	3.1721(−4)	2.7747(−4)	2.4499(−4)	2.1806(−4)	1.9547(−4)	1.7632(−4)	1.5994(−4)	1.4582(−4)
0.2	3.7343(−4)	3.2144(−4)	2.8019(−4)	2.4677(−4)	2.1926(−4)	1.9631(−4)	1.7694(−4)	1.6043(−4)	1.4623(−4)	1.3393(−4)
0.3	3.2384(−4)	2.8070(−4)	2.4627(−4)	2.1823(−4)	1.9501(−4)	1.7554(−4)	1.5902(−4)	1.4487(−4)	1.3264(−4)	1.2199(−4)
0.4	2.8225(−4)	2.4611(−4)	2.1715(−4)	1.9345(−4)	1.7374(−4)	1.5714(−4)	1.4299(−4)	1.3082(−4)	1.2026(−4)	1.1103(−4)
0.5	2.4783(−4)	2.1722(−4)	1.9262(−4)	1.7240(−4)	1.5552(−4)	1.4125(−4)	1.2905(−4)	1.1851(−4)	1.0934(−4)	1.0129(−4)
0.6	2.1900(−4)	1.9286(−4)	1.7179(−4)	1.5442(−4)	1.3987(−4)	1.2752(−4)	1.1692(−4)	1.0775(−4)	9.9742(−5)	9.2697(−5)
0.7	1.9457(−4)	1.7210(−4)	1.5394(−4)	1.3892(−4)	1.2630(−4)	1.1557(−4)	1.0633(−4)	9.8307(−5)	9.1283(−5)	8.5089(−5)
0.8	1.7397(−4)	1.5448(−4)	1.3870(−4)	1.2563(−4)	1.1461(−4)	1.0521(−4)	9.7107(−5)	9.0051(−5)	8.3859(−5)	7.8385(−5)
0.9	1.5668(−4)	1.3962(−4)	1.2578(−4)	1.1429(−4)	1.0459(−4)	9.6301(−5)	8.9134(−5)	8.2882(−5)	7.7384(−5)	7.2514(−5)
1	1.4137(−4)	1.2641(−4)	1.1426(−4)	1.0416(−4)	9.5611(−5)	8.8290(−5)	8.1949(−5)	7.6408(−5)	7.1525(−5)	6.7193(−5)

TABLE IV

The Exit Angular Fluxes $\psi_i(R,\mu)$ for $i = 1$–10

μ	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$	$i=8$	$i=9$	$i=10$
0	1.1146(−3)	3.4317(−4)	1.7392(−4)	1.0772(−4)	1.1525(−4)	5.8073(−5)	4.4676(−5)	3.5756(−5)	2.9444(−5)	2.9886(−5)
0.1	1.5864(−3)	4.5301(−4)	2.2419(−4)	1.3714(−4)	1.4787(−4)	7.2858(−5)	5.5763(−5)	4.4432(−5)	3.6442(−5)	3.7031(−5)
0.2	2.4017(−3)	5.9970(−4)	2.8704(−4)	1.7251(−4)	1.8566(−4)	8.9704(−5)	6.8195(−5)	5.4026(−5)	4.4088(−5)	4.4666(−5)
0.3	4.9489(−3)	8.0657(−4)	3.7054(−4)	2.1806(−4)	2.3364(−4)	1.1050(−4)	8.3362(−5)	6.5611(−5)	5.3236(−5)	5.3710(−5)
0.4	1.2386(−2)	1.0994(−3)	4.8210(−4)	2.7724(−4)	2.9545(−4)	1.3656(−4)	1.0217(−4)	7.9849(−5)	6.4390(−5)	6.4659(−5)
0.5	2.6436(−2)	1.5012(−3)	6.2888(−4)	3.5321(−4)	3.7446(−4)	1.6903(−4)	1.2541(−4)	9.7307(−5)	7.7979(−5)	7.7929(−5)
0.6	4.6527(−2)	2.0197(−3)	8.1612(−4)	4.4840(−4)	4.7327(−4)	2.0877(−4)	1.5366(−4)	1.1842(−4)	9.4325(−5)	9.3832(−5)
0.7	7.1061(−2)	2.6424(−3)	1.0452(−3)	5.6385(−4)	5.9301(−4)	2.5621(−4)	1.8722(−4)	1.4338(−4)	1.1359(−4)	1.1253(−4)
0.8	9.8380(−2)	3.3420(−3)	1.3129(−3)	6.9894(−4)	7.3295(−4)	3.1124(−4)	2.2603(−4)	1.7218(−4)	1.3576(−4)	1.3403(−4)
0.9	1.2713(−1)	4.0868(−3)	1.6124(−3)	8.5165(−4)	8.9074(−4)	3.7331(−4)	2.6973(−4)	2.0456(−4)	1.6065(−4)	1.5817(−4)
1	1.5630(−1)	4.8463(−3)	1.9351(−3)	1.0191(−3)	1.0630(−3)	4.4168(−4)	3.1785(−4)	2.4019(−4)	1.8803(−4)	1.8473(−4)

TABLE V

The Exit Angular Fluxes $\psi_i(R,\mu)$ for $i = 11$–20

μ	$i=11$	$i=12$	$i=13$	$i=14$	$i=15$	$i=16$	$i=17$	$i=18$	$i=19$	$i=20$
0	2.1534(−5)	1.8639(−5)	1.6352(−5)	1.4499(−5)	1.2971(−5)	1.1692(−5)	1.0610(−5)	9.6829(−6)	8.8825(−6)	8.1854(−6)
0.1	2.6462(−5)	2.2831(−5)	1.9968(−5)	1.7653(−5)	1.5748(−5)	1.4157(−5)	1.2812(−5)	1.1662(−5)	1.0671(−5)	9.8099(−6)
0.2	3.1729(−5)	2.7274(−5)	2.3770(−5)	2.0946(−5)	1.8627(−5)	1.6695(−5)	1.5066(−5)	1.3677(−5)	1.2482(−5)	1.1446(−5)
0.3	3.7922(−5)	3.2465(−5)	2.8187(−5)	2.4748(−5)	2.1934(−5)	1.9596(−5)	1.7630(−5)	1.5958(−5)	1.4523(−5)	1.3282(−5)
0.4	4.5360(−5)	3.8664(−5)	3.3432(−5)	2.9242(−5)	2.5823(−5)	2.2992(−5)	2.0618(−5)	1.8605(−5)	1.6882(−5)	1.5395(−5)
0.5	5.4309(−5)	4.6087(−5)	3.9685(−5)	3.4574(−5)	3.0419(−5)	2.6989(−5)	2.4122(−5)	2.1698(−5)	1.9629(−5)	1.7847(−5)
0.6	6.4972(−5)	5.4898(−5)	4.7079(−5)	4.0858(−5)	3.5817(−5)	3.1670(−5)	2.8212(−5)	2.5298(−5)	2.2817(−5)	2.0686(−5)
0.7	7.7453(−5)	6.5183(−5)	5.5688(−5)	4.8158(−5)	4.2073(−5)	3.7080(−5)	3.2930(−5)	2.9441(−5)	2.6477(−5)	2.3939(−5)
0.8	9.1751(−5)	7.6946(−5)	6.5518(−5)	5.6478(−5)	4.9192(−5)	4.3229(−5)	3.8284(−5)	3.4135(−5)	3.0620(−5)	2.7615(−5)
0.9	1.0777(−4)	9.0113(−5)	7.6512(−5)	6.5776(−5)	5.7142(−5)	5.0089(−5)	4.4252(−5)	3.9365(−5)	3.5232(−5)	3.1704(−5)
1	1.2538(−4)	1.0458(−4)	8.8584(−5)	7.5982(−5)	6.5865(−5)	5.7615(−5)	5.0797(−5)	4.5098(−5)	4.0285(−5)	3.6183(−5)

TABLE VI

The Angular Fluxes $\psi_i(z,\mu)$ for $z = L + \Delta/4$ and $i = 1$–10

μ	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$	$i=8$	$i=9$	$i=10$
−1	1.0471(−3)	4.7532(−4)	2.9442(−4)	2.0836(−4)	2.2548(−4)	1.3465(−4)	1.1058(−4)	9.3621(−5)	8.1008(−5)	8.4882(−5)
−0.8	1.4632(−3)	6.3596(−4)	3.8406(−4)	2.6721(−4)	2.8916(−4)	1.6854(−4)	1.3719(−4)	1.1526(−4)	9.9037(−5)	1.0339(−4)
−0.6	2.1471(−3)	8.8218(−4)	5.1690(−4)	3.5252(−4)	3.8060(−4)	2.1598(−4)	1.7403(−4)	1.4490(−4)	1.2351(−4)	1.2833(−4)
−0.4	3.3761(−3)	1.2884(−3)	7.2703(−4)	4.8356(−4)	5.1900(−4)	2.8558(−4)	2.2731(−4)	1.8724(−4)	1.5808(−4)	1.6323(−4)
−0.2	5.7980(−3)	1.9994(−3)	1.0739(−3)	6.9193(−4)	7.3619(−4)	3.9049(−4)	3.0636(−4)	2.4921(−4)	2.0807(−4)	2.1325(−4)
0	1.1621(−2)	3.3843(−3)	1.6919(−3)	1.0444(−3)	1.0971(−3)	5.5598(−4)	4.2861(−4)	3.4339(−4)	2.8291(−4)	2.8730(−4)
0.2	1.0604(−1)	6.1309(−3)	2.7893(−3)	1.6302(−3)	1.6888(−3)	8.1014(−4)	6.1246(−4)	4.8257(−4)	3.9185(−4)	3.9406(−4)
0.4	3.1493(−1)	8.9005(−3)	4.1886(−3)	2.4158(−3)	2.4442(−3)	1.1457(−3)	8.5297(−4)	6.6300(−4)	5.3192(−4)	5.3084(−4)
0.6	4.6327(−1)	1.0270(−2)	5.2044(−3)	3.1081(−3)	3.0428(−3)	1.4886(−3)	1.1061(−3)	8.5588(−4)	6.8292(−4)	6.7815(−4)
0.8	5.6269(−1)	1.0647(−2)	5.7110(−3)	3.5443(−3)	3.3742(−3)	1.7593(−3)	1.3194(−3)	1.0257(−3)	8.1998(−4)	8.1111(−4)
1	6.3224(−1)	1.0504(−2)	5.8661(−3)	3.7561(−3)	3.4995(−3)	1.9369(−3)	1.4720(−3)	1.1550(−3)	9.2922(−4)	9.1575(−4)

TABLE VII

The Angular Fluxes $\psi_i(z,\mu)$ for $z = L + \Delta/4$ and $i = 11$–20

μ	$i=11$	$i=12$	$i=13$	$i=14$	$i=15$	$i=16$	$i=17$	$i=18$	$i=19$	$i=20$
−1	6.4895(−5)	5.8213(−5)	5.2815(−5)	4.8333(−5)	4.4547(−5)	4.1304(−5)	3.8496(−5)	3.6043(−5)	3.3881(−5)	3.1962(−5)
−0.8	7.8313(−5)	6.9873(−5)	6.3073(−5)	5.7443(−5)	5.2700(−5)	4.8650(−5)	4.5153(−5)	4.2105(−5)	3.9427(−5)	3.7055(−5)
−0.6	9.6221(−5)	8.5334(−5)	7.6589(−5)	6.9376(−5)	6.3321(−5)	5.8170(−5)	5.3738(−5)	4.9888(−5)	4.6514(−5)	4.3537(−5)
−0.4	1.2098(−4)	1.0654(−4)	9.4989(−5)	8.5506(−5)	7.7585(−5)	7.0875(−5)	6.5128(−5)	6.0156(−5)	5.5817(−5)	5.2003(−5)
−0.2	1.5600(−4)	1.3629(−4)	1.2060(−4)	1.0779(−4)	9.7152(−5)	8.8191(−5)	8.0555(−5)	7.3982(−5)	6.8274(−5)	6.3278(−5)
0	2.0702(−4)	1.7916(−4)	1.5715(−4)	1.3931(−4)	1.2459(−4)	1.1228(−4)	1.0185(−4)	9.2930(−5)	8.5228(−5)	7.8524(−5)
0.2	2.7932(−4)	2.3930(−4)	2.0792(−4)	1.8270(−4)	1.6205(−4)	1.4491(−4)	1.3049(−4)	1.1823(−4)	1.0772(−4)	9.8623(−5)
0.4	3.7090(−4)	3.1505(−4)	2.7154(−4)	2.3679(−4)	2.0854(−4)	1.8522(−4)	1.6572(−4)	1.4924(−4)	1.3518(−4)	1.2307(−4)
0.6	4.7031(−4)	3.9743(−4)	3.4080(−4)	2.9574(−4)	2.5922(−4)	2.2918(−4)	2.0416(−4)	1.8308(−4)	1.6515(−4)	1.4977(−4)
0.8	5.6385(−4)	4.7583(−4)	4.0729(−4)	3.5271(−4)	3.0848(−4)	2.7212(−4)	2.4185(−4)	2.1638(−4)	1.9474(−4)	1.7619(−4)
1	6.4306(−4)	5.4362(−4)	4.6572(−4)	4.0342(−4)	3.5278(−4)	3.1106(−4)	2.7628(−4)	2.4699(−4)	2.2208(−4)	2.0073(−4)

TABLE VIII

The Angular Fluxes $\psi_i(z,\mu)$ for $z = L + \Delta/2$ and $i = 1$–10

μ	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$	$i=8$	$i=9$	$i=10$
−1	4.6637(−4)	2.2300(−4)	1.4098(−4)	1.0055(−4)	1.0980(−4)	6.5688(−5)	5.3896(−5)	4.5600(−5)	3.9433(−5)	4.1400(−5)
−0.8	6.5719(−4)	3.0006(−4)	1.8405(−4)	1.2860(−4)	1.4081(−4)	8.1781(−5)	6.6465(−5)	5.5781(−5)	4.7897(−5)	5.0105(−5)
−0.6	9.7087(−4)	4.1608(−4)	2.4616(−4)	1.6813(−4)	1.8419(−4)	1.0381(−4)	8.3510(−5)	6.9474(−5)	5.9192(−5)	6.1650(−5)
−0.4	1.5206(−3)	5.9792(−4)	3.3951(−4)	2.2623(−4)	2.4721(−4)	1.3507(−4)	1.0743(−4)	8.8500(−5)	7.4746(−5)	7.7431(−5)
−0.2	2.5630(−3)	9.0213(−4)	4.8879(−4)	3.1638(−4)	3.4365(−4)	1.8139(−4)	1.4239(−4)	1.1596(−4)	9.6954(−5)	9.9770(−5)
0	4.7967(−3)	1.4586(−3)	7.4368(−4)	4.6406(−4)	4.9928(−4)	2.5304(−4)	1.9558(−4)	1.5716(−4)	1.2986(−4)	1.3254(−4)
0.2	1.7611(−2)	2.5687(−3)	1.2010(−3)	7.1509(−4)	7.5999(−4)	3.6685(−4)	2.7848(−4)	2.2033(−4)	1.7961(−4)	1.8155(−4)
0.4	1.0153(−1)	4.5595(−3)	1.9722(−3)	1.1178(−3)	1.1749(−3)	5.3863(−4)	4.0154(−4)	3.1281(−4)	2.5157(−4)	2.5190(−4)
0.6	2.1428(−1)	6.8622(−3)	2.9949(−3)	1.6625(−3)	1.7253(−3)	7.6632(−4)	5.6338(−4)	4.3358(−4)	3.4496(−4)	3.4299(−4)
0.8	3.1513(−1)	8.7448(−3)	4.0077(−3)	2.2513(−3)	2.2971(−3)	1.0242(−3)	7.4781(−4)	5.7151(−4)	4.5170(−4)	4.4733(−4)
1	3.9787(−1)	1.0026(−2)	4.8357(−3)	2.7862(−3)	2.7919(−3)	1.2807(−3)	9.3504(−4)	7.1320(−4)	5.6215(−4)	5.5567(−4)

TABLE IX

The Angular Fluxes $\psi_i(z,\mu)$ for $z = L + \Delta/2$ and $i = 11$–20

μ	$i=11$	$i=12$	$i=13$	$i=14$	$i=15$	$i=16$	$i=17$	$i=18$	$i=19$	$i=20$
−1	3.1587(−5)	2.8323(−5)	2.5691(−5)	2.3508(−5)	2.1664(−5)	2.0087(−5)	1.8722(−5)	1.7530(−5)	1.6480(−5)	1.5549(−5)
−0.8	3.7875(−5)	3.3784(−5)	3.0493(−5)	2.7771(−5)	2.5480(−5)	2.3525(−5)	2.1839(−5)	2.0369(−5)	1.9079(−5)	1.7937(−5)
−0.6	4.6145(−5)	4.0923(−5)	3.6736(−5)	3.3286(−5)	3.0392(−5)	2.7931(−5)	2.5815(−5)	2.3977(−5)	2.2368(−5)	2.0947(−5)
−0.4	5.7338(−5)	5.0525(−5)	4.5082(−5)	4.0615(−5)	3.6885(−5)	3.3725(−5)	3.1018(−5)	2.8677(−5)	2.6633(−5)	2.4835(−5)
−0.2	7.3005(−5)	6.3863(−5)	5.6592(−5)	5.0657(−5)	4.5725(−5)	4.1568(−5)	3.8023(−5)	3.4969(−5)	3.2315(−5)	2.9990(−5)
0	9.5688(−5)	8.3007(−5)	7.2978(−5)	6.4843(−5)	5.8122(−5)	5.2492(−5)	4.7716(−5)	4.3624(−5)	4.0085(−5)	3.7000(−5)
0.2	1.2911(−4)	1.1094(−4)	9.6668(−5)	8.5174(−5)	7.5748(−5)	6.7903(−5)	6.1293(−5)	5.5663(−5)	5.0823(−5)	4.6628(−5)
0.4	1.7643(−4)	1.5016(−4)	1.2967(−4)	1.1329(−4)	9.9953(−5)	8.8931(−5)	7.9705(−5)	7.1895(−5)	6.5220(−5)	5.9467(−5)
0.6	2.3717(−4)	2.0027(−4)	1.7167(−4)	1.4894(−4)	1.3053(−4)	1.1541(−4)	1.0281(−4)	9.2203(−5)	8.3179(−5)	7.5437(−5)
0.8	3.0662(−4)	2.5757(−4)	2.1968(−4)	1.8968(−4)	1.6548(−4)	1.4566(−4)	1.2921(−4)	1.1540(−4)	1.0369(−4)	9.3668(−5)
1	3.7912(−4)	3.1754(−4)	2.7005(−4)	2.3251(−4)	2.0228(−4)	1.7757(−4)	1.5710(−4)	1.3994(−4)	1.2542(−4)	1.1302(−4)

TABLE X

The Angular Fluxes $\psi_i(z,\mu)$ for $z = L + 3\Delta/4$ and $i = 1$–10

μ	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$	$i=8$	$i=9$	$i=10$
−1	1.7860(−4)	9.3645(−5)	6.2715(−5)	4.6495(−5)	4.9553(−5)	3.1722(−5)	2.6391(−5)	2.2549(−5)	1.9639(−5)	2.0604(−5)
−0.8	2.5809(−4)	1.3050(−4)	8.4778(−5)	6.1291(−5)	6.5936(−5)	4.0308(−5)	3.3052(−5)	2.7903(−5)	2.4057(−5)	2.5181(−5)
−0.6	3.9570(−4)	1.8906(−4)	1.1763(−4)	8.2318(−5)	8.9400(−5)	5.1863(−5)	4.1893(−5)	3.4944(−5)	2.9829(−5)	3.1112(−5)
−0.4	6.5510(−4)	2.8481(−4)	1.6683(−4)	1.1233(−4)	1.2293(−4)	6.7677(−5)	5.3904(−5)	4.4458(−5)	3.7591(−5)	3.9017(−5)
−0.2	1.1985(−3)	4.4257(−4)	2.4142(−4)	1.5663(−4)	1.7143(−4)	9.0429(−5)	7.1062(−5)	5.7945(−5)	4.8510(−5)	5.0033(−5)
0	2.2618(−3)	7.0354(−4)	3.6161(−4)	2.2667(−4)	2.4631(−4)	1.2494(−4)	9.6756(−5)	7.7905(−5)	6.4499(−5)	6.6013(−5)
0.2	5.4530(−3)	1.2061(−3)	5.7458(−4)	3.4520(−4)	3.7093(−4)	1.7989(−4)	1.3697(−4)	1.0869(−4)	8.8846(−5)	9.0085(−5)
0.4	3.4292(−2)	2.2024(−3)	9.5455(−4)	5.4592(−4)	5.7968(−4)	2.6734(−4)	1.9988(−4)	1.5615(−4)	1.2590(−4)	1.2636(−4)
0.6	9.9505(−2)	3.7912(−3)	1.5582(−3)	8.5481(−4)	8.9957(−4)	3.9606(−4)	2.9133(−4)	2.2441(−4)	1.7871(−4)	1.7778(−4)
0.8	1.7614(−1)	5.6253(−3)	2.3367(−3)	1.2609(−3)	1.3159(−3)	5.6355(−4)	4.0973(−4)	3.1239(−4)	2.4652(−4)	2.4372(−4)
1	2.4965(−1)	7.3271(−3)	3.1650(−3)	1.7184(−3)	1.7754(−3)	7.5725(−4)	5.4699(−4)	4.1448(−4)	3.2522(−4)	3.2042(−4)

TABLE XI

The Angular Fluxes $\psi_i(z,\mu)$ for $z = L + 3\Delta/4$ and $i = 11$–20

[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
−1	1.5874(−5)	1.4278(−5)	1.2982(−5)	1.1902(−5)	1.0986(−5)	1.0199(−5)	9.5172(−6)	8.9201(−6)	8.3934(−6)	7.9254(−6)
−0.8	1.9123(−5)	1.7087(−5)	1.5444(−5)	1.4082(−5)	1.2934(−5)	1.1952(−5)	1.1105(−5)	1.0365(−5)	9.7154(−6)	9.1398(−6)
−0.6	2.3322(−5)	2.0704(−5)	1.8603(−5)	1.6870(−5)	1.5416(−5)	1.4178(−5)	1.3113(−5)	1.2188(−5)	1.1377(−5)	1.0662(−5)
−0.4	2.8907(−5)	2.5495(−5)	2.2768(−5)	2.0530(−5)	1.8660(−5)	1.7075(−5)	1.5717(−5)	1.4542(−5)	1.3515(−5)	1.2612(−5)
−0.2	3.6635(−5)	3.2082(−5)	2.8460(−5)	2.5502(−5)	2.3043(−5)	2.0969(−5)	1.9199(−5)	1.7674(−5)	1.6347(−5)	1.5184(−5)
0	4.7726(−5)	4.1465(−5)	3.6510(−5)	3.2488(−5)	2.9161(−5)	2.6371(−5)	2.4003(−5)	2.1971(−5)	2.0213(−5)	1.8678(−5)
0.2	6.4217(−5)	5.5291(−5)	4.8272(−5)	4.2612(−5)	3.7962(−5)	3.4087(−5)	3.0817(−5)	2.8029(−5)	2.5628(−5)	2.3544(−5)
0.4	8.8735(−5)	7.5661(−5)	6.5454(−5)	5.7283(−5)	5.0621(−5)	4.5106(−5)	4.0483(−5)	3.6565(−5)	3.3212(−5)	3.0318(−5)
0.6	1.2312(−4)	1.0405(−4)	8.9251(−5)	7.7488(−5)	6.7959(−5)	6.0122(−5)	5.3591(−5)	4.8087(−5)	4.3403(−5)	3.9382(−5)
0.8	1.6695(−4)	1.4013(−4)	1.1943(−4)	1.0304(−4)	8.9839(−5)	7.9028(−5)	7.0060(−5)	6.2534(−5)	5.6156(−5)	5.0702(−5)
1	2.1784(−4)	1.8202(−4)	1.5445(−4)	1.3271(−4)	1.1524(−4)	1.0098(−4)	8.9177(−5)	7.9305(−5)	7.0960(−5)	6.3842(−5)

We have found that these expressions are significant improvements especially as $\mu \to 0$ over the usual (and simpler) expressions given by Eqs. (55) and (95).

In addition we show in Table XII converged results for the group fluxes

$$\phi_i(z) = \int_{-1}^{1} \psi_i(z, \mu)\, d\mu \tag{126}$$

which, for $z = L$ and $z = R$, can be expressed by using Eqs. (55) and (120) as

$$\phi_i(L) = \delta_{i,1} + a_{i,0} \tag{127a}$$

and

$$\phi_i(R) = \delta_{i,1} E_2(\Delta_i) + b_{i,0}, \tag{127b}$$

where, in general, $E_n(x)$ denotes exponential-integral functions. For $z \in (L, R)$ we use Eqs. (95) and (120) in Eq. (126) to obtain

$$\phi_i(z) = \delta_{i,1} E_2[\sigma_i(z - L)] + c_{i,0}(z) + d_{i,0}(z). \tag{127c}$$

Finally, we list in Table XIII our converged results for the group albedos

$$A_i^* = 2 \int_0^1 \mu \psi_i(L, -\mu)\, d\mu \tag{128a}$$

TABLE XII

The Group Fluxes $\phi_i(z)$

i	$z = L$	$z = L + \Delta/4$	$z = L + \Delta/2$	$z = L + 3\Delta/4$	$z = R$
1	1.0117	3.5594(−1)	1.7050(−1)	8.7855(−2)	4.6716(−2)
2	3.9657(−3)	9.9306(−3)	6.2916(−3)	3.6317(−3)	1.9091(−3)
3	2.0883(−3)	5.0739(−3)	2.9221(−3)	1.5888(−3)	7.8103(−4)
4	1.3228(−3)	3.1027(−3)	1.6910(−3)	8.9818(−4)	4.2724(−4)
5	1.3007(−3)	3.0869(−3)	1.7578(−3)	9.4748(−4)	4.5042(−4)
6	7.0807(−4)	1.5694(−3)	8.2100(−4)	4.3208(−4)	1.9734(−4)
7	5.5208(−4)	1.1895(−3)	6.1336(−4)	3.2196(−4)	1.4496(−4)
8	4.4579(−4)	9.3681(−4)	4.7874(−4)	2.5096(−4)	1.1154(−4)
9	3.6946(−4)	7.5937(−4)	3.8576(−4)	2.0207(−4)	8.8744(−5)
10	3.6961(−4)	7.6078(−4)	3.8710(−4)	2.0269(−4)	8.8221(−5)
11	2.7104(−4)	5.3840(−4)	2.7184(−4)	1.4228(−4)	6.1011(−5)
12	2.3568(−4)	4.6029(−4)	2.3188(−4)	1.2133(−4)	5.1520(−5)
13	2.0748(−4)	3.9910(−4)	2.0072(−4)	1.0500(−4)	4.4159(−5)
14	1.8450(−4)	3.5000(−4)	1.7580(−4)	9.1954(−5)	3.8308(−5)
15	1.6545(−4)	3.0989(−4)	1.5550(−4)	8.1325(−5)	3.3569(−5)
16	1.4946(−4)	2.7666(−4)	1.3871(−4)	7.2537(−5)	2.9673(−5)
17	1.3588(−4)	2.4877(−4)	1.2465(−4)	6.5175(−5)	2.6427(−5)
18	1.2423(−4)	2.2511(−4)	1.1273(−4)	5.8941(−5)	2.3692(−5)
19	1.1415(−4)	2.0486(−4)	1.0254(−4)	5.3609(−5)	2.1365(−5)
20	1.0536(−4)	1.8737(−4)	9.3748(−5)	4.9008(−5)	1.9367(−5)

TABLE XIII

A_i^* and B_i^*

	Present work		DTF69	
i	A_i^*	B_i^*	A_i^*	B_i^*
1	6.4394(−3)	7.3100(−2)	6.4399(−3)	7.2983(−2)
2	2.4468(−3)	2.6667(−3)	2.4467(−3)	2.6646(−3)
3	1.3718(−3)	1.0693(−3)	1.3719(−3)	1.0678(−3)
4	9.0655(−4)	5.7560(−4)	9.0668(−4)	5.7472(−4)
5	9.1002(−4)	6.0465(−4)	9.1011(−4)	6.0380(−4)
6	5.1696(−4)	2.5976(−4)	5.1708(−4)	2.5935(−4)
7	4.1123(−4)	1.8942(−4)	4.1134(−4)	1.8912(−4)
8	3.3795(−4)	1.4482(−4)	3.3805(−4)	1.4458(−4)
9	2.8449(−4)	1.1456(−4)	2.8459(−4)	1.1437(−4)
10	2.8836(−4)	1.1340(−4)	2.8844(−4)	1.1321(−4)
11	2.1483(−4)	7.7912(−5)	2.1492(−4)	7.7785(−4)
12	1.8886(−4)	6.5506(−5)	1.8894(−4)	6.5399(−5)
13	1.6797(−4)	5.5914(−5)	1.6805(−4)	5.5822(−5)
14	1.5080(−4)	4.8312(−5)	1.5088(−4)	4.8233(−5)
15	1.3645(−4)	4.2175(−5)	1.3652(−4)	4.2106(−5)
16	1.2430(−4)	3.7144(−5)	1.2437(−4)	3.7082(−5)
17	1.1390(−4)	3.2964(−5)	1.1397(−4)	3.2909(−5)
18	1.0491(−4)	2.9452(−5)	1.0497(−4)	2.9402(−5)
19	9.7071(−5)	2.6472(−5)	9.7135(−5)	2.6427(−5)
20	9.0188(−5)	2.3920(−5)	9.0250(−5)	2.3879(−5)

and the group transmission factors

$$B_i^* = 2\int_0^1 \mu\psi_i(R,\mu)\,d\mu. \tag{128b}$$

If we use Eqs. (55) and (120) in Eqs. (128) we find

$$A_i^* = a_{i,0} + \tfrac{1}{3}a_{i,1} \tag{129a}$$

and

$$B_i^* = 2\delta_{i,1}E_3(\Delta_i) + b_{i,0} + \tfrac{1}{3}b_{i,1}. \tag{129b}$$

All our numerical results are accurate, we believe, to within ± 1 in the last digit shown. In Table XIII we also show the results of a calculation by Renken [14] who used DTF69, a discrete-ordinates code [15], with 40 space points and eight directions for each half range of μ. We observe here as in the isotropic scattering case [2] what we believe to be a slight deterioration in the DTF69 results for increasing absorption (as the group number increases).

Regarding the convergence of our method, we have found that to establish

$\psi_l(L, -\mu)$ and $\psi_l(R, \mu)$ accurate to what we believe to be five significant figures for all μ required in this case $N = 20$; for the interior angular fluxes and the integrated quantities $\phi_l(z)$, A_l^*, and B_l^* we have found that $N = 15$ was sufficient to obtain five figures of accuracy.

Finally, we would like to mention that we have also generated numerical results for the 19-group problem considered in [2], but generalized to include anisotropic-scattering effects of the Klein–Nishina differential scattering cross section. A set of P_5 multigroup transfer cross sections was provided by Renken [14]. Due to the truncation of the Legendre expansion of the cross section, however, the resulting multigroup transfer cross sections turned out to be negative for some values of the scattering angle. Thus, the familiar and challenging question of how to deal with the solution of a strictly nonphysical problem was encountered [16]. From a mathematical point of view a transport equation based on cross sections that can be negative is a perfectly valid candidate for study and can clearly yield a solution that can be negative. One can (as we did) solve such a problem and accept, at least on a mathematical basis, the solution—positive or otherwise. We note that in comparing our results for the considered problem with P_5 scattering with those obtained by Renken [14], with and without the use of the negative-flux fix-up option [15] in the DTF69 code, we found excellent agreement with Renken's results only when he did not use the negative-flux fix-up option. A separate question is what is the relationship of the solution obtained in this way to the physically correct solution that satisfies the transport equation for which the Klein–Nishina cross section has not been truncated.

6. Conclusions

We conclude from our studies that the F_N method is capable of producing accurate results for the considered multigroup model. The most interesting aspect of the method seems to be the capability of finding the reflected and transmitted angular fluxes for a given group by using only the boundary data and established emerging fluxes for preceding groups. This feature of the F_N method is a particularly attractive one for shielding calculations, where frequently the interior angular fluxes are not of primary interest. For most situations the results deduced from the method of discrete ordinates are clearly adequate—especially when we consider the magnitude of the uncertainties associated with the input data. For strong absorption and/or optically thick slabs, however, increased computer time will be required by strictly numerical methods to achieve a desired degree of accuracy—a characteristic not shared by the F_N method.

Appendix A: Alternative Expressions for $\Phi_{j,0}(s/\sigma_j)$

We have mentioned in Section 2 that an alternative expression to Eq. (37) is needed to compute $\Phi_{1,0}(s/\sigma_1)$ in the event that $G_1(s, s) = 0$ for some s. The same is true, in general, for the $\Phi_{j,0}(s/\sigma_j)$ given by Eq. (51) when $G_j(s, s) = 0$. It is clear that Eq. (37) is only a special case of Eq. (51), and thus we now proceed to derive an alternative expression to the latter. If we set $n = 0$ in Eq. (7) and use Eq. (49) we obtain, for $s \notin [-1, 1]$,

$$\sigma_j \Lambda_j(s)\, \Phi_{j,0}(s/\sigma_j) = s \int_{-1}^{1} \mu B_j(\mu, s) \frac{d\mu}{\mu - s} - s \sum_{l=0}^{\mathscr{L}} \sigma_{jj}(l)\, D_{j,l}(s)\, Q_l(s) + s \sum_{k=1}^{j-1} \sum_{l=0}^{\mathscr{L}} (-1)^l \sigma_{jk}(l)\, \Phi_{k,l}(s/\sigma_j)\, Q_l(s), \tag{A1}$$

where $Q_l(s)$ denote the Legendre functions of the second kind [17], i.e.,

$$(2l + 1)\, sQ_l(s) = (l + 1)\, Q_{l+1}(s) + lQ_{l-1}(s) + \delta_{0,l} \tag{A2}$$

with

$$Q_0(s) = \tfrac{1}{2} \log((s + 1)/(s - 1)), \qquad s \notin [-1, 1], \tag{A3}$$

or

$$Q_0(\nu) = \tfrac{1}{2} \log((1 + \nu)/(1 - \nu)), \qquad \nu \in [-1, 1]. \tag{A4}$$

If we demonstrate that $\Lambda_j(s)$ and $G_j(s, s)$ do not have common zeros for $s \notin [-1, 1]$ it is clear that we can divide Eq. (A1) by $\Lambda_j(s)$ to obtain the desired alternative formula for $\Phi_{j,0}(s/\sigma_j)$, $s \notin [-1, 1]$, in the event that $G_j(s, s) = 0$. First we use the summation formulas given by Inönü [18] to write $\Lambda_j(s)$ and $G_j(s, s)$ in the convenient forms

$$\Lambda_j(s) = (\mathscr{L} + 1)[Q_{\mathscr{L}}(s)\, g_{j,\mathscr{L}+1}(s) - Q_{\mathscr{L}+1}(s)\, g_{j,\mathscr{L}}(s)] \tag{A5}$$

and

$$G_j(s, s) = (c_j s)^{-1}(\mathscr{L} + 1)[P_{\mathscr{L}+1}(s)\, g_{j,\mathscr{L}}(s) - P_{\mathscr{L}}(s)\, g_{j,\mathscr{L}+1}(s)]. \tag{A6}$$

It is easy to show that the following identity holds:

$$(\mathscr{L} + 1)[P_{\mathscr{L}+1}(s)\, Q_{\mathscr{L}}(s) - P_{\mathscr{L}}(s)\, Q_{\mathscr{L}+1}(s)] = 1. \tag{A7}$$

We now multiply Eq. (A5) by $P_{\mathscr{L}+1}(s)$ and use Eqs. (A6) and (A7) to obtain

$$P_{\mathscr{L}+1}(s)\, \Lambda_j(s) = g_{j,\mathscr{L}+1}(s) - c_j s Q_{\mathscr{L}+1}(s)\, G_j(s, s). \tag{A8}$$

By contradiction, if we suppose that there exists some $s^* \notin [-1, 1]$ such that $\Lambda_j(s^*) = G_j(s^*, s^*) = 0$ we conclude immediately from Eq. (A8) that $g_{j,\mathscr{L}+1}(s^*) = 0$. From Eq. (A6), however, we see that this would require $g_{j,\mathscr{L}}(s^*) = 0$. Clearly, this is not possible; otherwise from Eq. (8) we would have $g_{j,l}(s^*) = 0$ for all l. We must then conclude that there is no such s^*.

In order to obtain an alternative formula for $\Phi_{j,0}(\nu/\sigma_j)$, $\nu \in [-1, 1]$, we let s approach the branch cut to find that Eq. (A1) yields, for $\nu \in [-1, 1]$,

$$\sigma_j c_j G_j(\nu, \nu)\, \Phi_{j,0}(\nu/\sigma_j) = 2\nu B_j(\nu, \nu) + \sum_{l=0}^{\mathscr{L}} \sigma_{jj}(l)\, D_{j,l}(\nu)\, P_l(\nu) - \sum_{k=1}^{j-1} \sum_{l=0}^{\mathscr{L}} (-1)^l \times \sigma_{jk}(l)\, \Phi_{k,l}(\nu/\sigma_j)\, P_l(\nu) \tag{A9}$$

and

$$\sigma_j \lambda_j(\nu)\, \Phi_{j,0}(\nu/\sigma_j) = \nu P \int_{-1}^{1} \mu B_j(\mu, \nu) \frac{d\mu}{\mu - \nu} - \nu \sum_{l=0}^{\mathscr{L}} \sigma_{jj}(l)\, D_{j,l}(\nu)\, Q_l(\nu) + \nu \sum_{k=1}^{j-1} \sum_{l=0}^{\mathscr{L}} (-1)^l \sigma_{jk}(l)\, \Phi_{k,l}(\nu/\sigma_j)\, Q_l(\nu). \tag{A10}$$

We can use Eqs. (A9) and (A10) to show that, for $\nu \in [-1, 1]$ and $G_j(\nu, \nu) = 0$,

$$\sigma_j \lambda_j(\nu)\, \Phi_{j,0}(\nu/\sigma_j) = 2\nu B_j(\nu, \nu) + \nu \int_{-1}^{1} \mu \left[\frac{B_j(\mu, \nu) - B_j(\nu, \nu)}{\mu - \nu}\right] d\mu + \nu \sum_{l=1}^{\mathscr{L}} \sigma_{jj}(l)\, D_{j,l}(\nu)\, \Gamma_l(\nu) - \nu \sum_{k=1}^{j-1} \sum_{l=1}^{\mathscr{L}} (-1)^l \sigma_{jk}(l)\, \Phi_{k,l}(\nu/\sigma_j)\, \Gamma_l(\nu), \tag{A11}$$

where the polynomials $\Gamma_l(\nu)$ can be generated with the recursion formula [12], for

$$(2l + 1)\, \nu \Gamma_l(\nu) = -\delta_{l,0} + (l + 1)\, \Gamma_{l+1}(\nu) + l\Gamma_{l-1}(\nu), \tag{A12}$$

where

$$\Gamma_0(\nu) = 0. \tag{A13}$$

Again, if we demonstrate that $\lambda_j(\nu)$ and $G_j(\nu, \nu)$ do not have common zeros for $\nu \in [-1, 1]$ we can divide Eq. (A11) by $\lambda_j(\nu)$ to obtain the desired alternative formula for $\Phi_{j,0}(\nu/\sigma_j)$, $\nu \in [-1, 1]$, when $G_j(\nu, \nu) = 0$. We let s approach the branch cut to obtain from Eq. (A5)

$$\lambda_j(\nu) = (\mathscr{L} + 1)[Q_{\mathscr{L}}(\nu)\, g_{j,\mathscr{L}+1}(\nu) - Q_{\mathscr{L}+1}(\nu)\, g_{j,\mathscr{L}}(\nu)] \tag{A14}$$

and

$$G_j(\nu, \nu) = (c_j \nu)^{-1} (\mathscr{L} + 1)[P_{\mathscr{L}+1}(\nu)\, g_{j,\mathscr{L}}(\nu) - P_{\mathscr{L}}(\nu)\, g_{j,\mathscr{L}+1}(\nu)]. \tag{A15}$$

Of course Eq. (A7) is still valid for $\nu \in [-1, 1]$,

$$(\mathscr{L} + 1)[P_{\mathscr{L}+1}(\nu)\, Q_{\mathscr{L}}(\nu) - P_{\mathscr{L}}(\nu)\, Q_{\mathscr{L}+1}(\nu)] = 1, \tag{A16}$$

and Eq. (A8) yields

$$P_{\mathscr{L}+1}(\nu)\,\lambda_j(\nu) = g_{j,\mathscr{L}+1}(\nu) - c_j \nu Q_{\mathscr{L}+1}(\nu)\,G_j(\nu,\nu). \tag{A17}$$

By contradiction, if we suppose that there exists some $\nu^* \in [-1, 1]$ such that $\lambda_j(\nu^*) = G_j(\nu^*, \nu^*) = 0$ we see from Eq. (A17) that we must have $g_{j,\mathscr{L}+1}(\nu^*) = 0$. As before, the possibility that $g_{j,\mathscr{L}}(\nu^*) = 0$ has to be ruled out, and thus $G_j(\nu^*, \nu^*) = 0$ would require $P_{\mathscr{L}+1}(\nu^*) = 0$ in Eq. (A15). At the same time $\lambda_j(\nu^*) = 0$ would require $Q_{\mathscr{L}+1}(\nu^*) = 0$ in Eq. (A14). But $P_{\mathscr{L}+1}(\nu^*)$ and $Q_{\mathscr{L}+1}(\nu^*)$ cannot be zero simultaneously, otherwise Eq. (A16) would be violated. We must then conclude that there is no such ν^*. We note that this result implies that there are no discrete eigenvalues embedded in the continuum.

APPENDIX B: RECURSIVE RELATIONS

The functions $A_{i,\alpha}(\xi)$ defined by Eq. (60) can be shown to satisfy, for $\alpha \geqslant 0$, the recursive relation

$$\alpha A_{i,\alpha-1}(\xi) + (2\alpha + 1)(2\xi + 1)\,A_{i,\alpha}(\xi) + (\alpha + 1)\,A_{i,\alpha+1}(\xi) = 2(2\alpha + 1) \sum_{l=0}^{\mathscr{L}} (-1)^l \beta_{ii}(l)\, g_{i,l}(\xi)\, T_{\alpha,l}, \tag{B1}$$

where for forward recursion the required initial value can be computed from

$$A_{i,0}(\xi) = \sum_{l=0}^{\mathscr{L}} (-1)^l \beta_{ii}(l)\, g_{i,l}(\xi)\, C_l(\xi). \tag{B2}$$

Here the functions $C_l(\xi)$ can be found from

$$lC_{l-1}(\xi) + (2l + 1)\,\xi C_l(\xi) + (l + 1)\,C_{l+1}(\xi) = (2l + 1)\,T_{0,l} \tag{B3}$$

with

$$C_0(\xi) = 1 - \xi \log(1 + (1/\xi)). \tag{B4}$$

We recall from Sections 3 and 4 that the functions $A_{i,\alpha}(\xi)$ are required for real $\xi \notin [-1, 0)$. We have found that the use of Eq. (B1) in the forward direction is stable only for $\xi \in (-1, 0]$, and thus an alternative procedure is desired. Using the Christoffel–Darboux formula [17] for the Legendre polynomials, we have deduced the alternative recursion relation

$$P_{\alpha+1}(2\xi + 1)\,A_{i,\alpha}(\xi) + P_\alpha(2\xi + 1)\,A_{i,\alpha+1}(\xi) = (-1)^\alpha (2/(\alpha + 1))\,\Gamma_{i,\alpha}(\xi), \tag{B5}$$

where

$$\Gamma_{i,0}(\xi) = \sum_{l=0}^{\mathscr{L}} (-1)^l \beta_{ii}(l)\, g_{i,l}(\xi)\, T_{0,l}, \tag{B6}$$

and, for $1 \leqslant \alpha \leqslant \mathscr{L} + 1$,

$$\Gamma_{i,\alpha}(\xi) = \Gamma_{i,\alpha-1}(\xi) + (-1)^{\alpha}(2\alpha + 1) P_{\alpha}(2\xi + 1) \sum_{l=0}^{\mathscr{L}} (-1)^{l} \beta_{il}(l)\, g_{i,l}(\xi)\, T_{\alpha,l}. \quad \text{(B7)}$$

Since $T_{\alpha,l} = 0$ for $\alpha > l + 1$, we see that Eq. (B7) yields

$$\Gamma_{i,\alpha}(\xi) = \Gamma_{i,\mathscr{L}+1}(\xi), \qquad \alpha > \mathscr{L} + 1. \quad \text{(B8)}$$

We have found that backward recursion of Eq. (B5) in the manner suggested by Miller [19] is stable for real $\xi \notin [-1, 0)$. As discussed before in [2], however, such a scheme can be time consuming for ξ close to the transition points -1 and 0, and for this reason we have actually used forward recursion of Eq. (B1) for $\xi \in [-1.001, -1) \cup [0, 0.001]$ without losing too many significant figures. For other ξ we used backward recursion of Eq. (B5). The functions $B_{i,\alpha}(\xi)$ defined by Eqs. (61) and required for $\xi \in P_i$ can be deduced from the recursive relation

$$-\alpha B_{i,\alpha-1}(\xi) + (2\alpha + 1)(2\xi - 1) B_{i,\alpha}(\xi) - (\alpha + 1) B_{i,\alpha+1}(\xi)$$
$$= 2(2\alpha + 1)\, c_i \sum_{l=0}^{\mathscr{L}} \beta_{il}(l)\, g_{i,l}(\xi)\, T_{\alpha,l}, \quad \text{(B9)}$$

with

$$B_{i,0}(\xi) = 2(1 - c_i) + c_i A_{i,0}(\xi). \quad \text{(B10)}$$

Forward recursion of Eq. (B9) can be used efficiently to generate $B_{i,\alpha}(\xi)$ for $\xi \in [0, 1]$. For $\xi = \nu_{i,m}$ it is easy to see that $B_{i,\alpha}(\nu_{i,m}) = -c_i A_{i,\alpha}(-\nu_{i,m})$, and thus, the recursion relations developed for $A_{i,\alpha}(\xi)$ can be readily used to establish $B_{i,\alpha}(\nu_{i,m})$. We note that the strategy adopted here yielded $A_{i,\alpha}(\xi)$ and $B_{i,\alpha}(\xi)$ accurate to at least 13 significant figures for α up to 40 (working in double precision with an IBM 370/165 machine).

We now turn our attention to the constants $T_{\alpha,l}$ defined by Eq. (70). We note that the $T_{\alpha,l}$ are a special case ($m = 0$) of the more general $T^{m}_{\alpha,l}$ considered by Devaux *et al.* [10], and thus we write, for $\alpha \geqslant 0$ and $l \geqslant 0$,

$$T_{\alpha,l+1} = (2l + 1)/[2(l + 1)]\,[(\alpha/(2\alpha + 1))\, T_{\alpha-1,l} + T_{\alpha,l} + ((\alpha + 1)/(2\alpha + 1))\, T_{\alpha+1,l}]$$
$$- (l/(l + 1))\, T_{\alpha,l-1}. \quad \text{(B11)}$$

In this equation α runs from $\alpha = 0$ to $l + 2$ (note that $T_{\beta,n} = 0$ for $\beta > n + 1$) for each l, from $l = 0$ to $\mathscr{L} - 1$. To initiate our calculation we use

$$T_{0,0} = \tfrac{1}{2} \quad \text{(B12)}$$

and

$$T_{1,0} = \tfrac{1}{6}. \quad \text{(B13)}$$

Finally, the polynomials $G_{i,\alpha}(\xi)$ defined by Eq. (73) and required for $\xi \in [0, 1]$ can be computed efficiently by forward recursion from

$$\alpha G_{i,\alpha-1}(\xi) - (2\alpha+1)(2\xi-1)\, G_{i,\alpha}(\xi) + (\alpha+1)\, G_{i,\alpha+1}(\xi) = 2(2\alpha+1) \sum_{l=0}^{\mathscr{L}} \beta_{ii}(l)\, g_{i,l}(\xi)\, T_{\alpha,l}, \tag{B14}$$

with

$$G_{i,0}(\xi) = 0. \tag{B15}$$

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References

1. C. E. Siewert and P. Benoist, *Nucl. Sci. Eng.* **78** (1981), 311.
2. R. D. M. Garcia and C. E. Siewert, *Nucl. Sci. Eng.* **78** (1981), 315.
3. S. Chandrasekhar, "Radiative Transfer," Oxford Univ. Press, London, 1950.
4. J. R. Mika, *Nucl. Sci. Eng.* **11** (1961), 415.
5. N. I. Muskhelishvili, "Singular Integral Equations," Noordhoff, Groningen, 1953.
6. R. L. Bowden, F. J. McCrosson, and E. A. Rhodes, *J. Math. Phys.* **9** (1968), 753.
7. C. E. Siewert, *Astrophys. Space Sci.* **58** (1978), 131.
8. C. E. Siewert and P. Benoist, *Nucl. Sci. Eng.* **69** (1979), 156.
9. P. Grandjean and C. E. Siewert, *Nucl. Sci. Eng.* **69** (1979), 161.
10. C. Devaux, C. E. Siewert, and Y. L. Yuan, *Astrophys. J.* **253** (1982), 773.
11. L. G. Henyey and J. L. Greenstein, *Astrophys. J.* **93** (1941), 70.
12. C. E. Siewert, *J. Math. Phys.* **21** (1980), 2468.
13. R. D. M. Garcia and C. E. Siewert, *Nucl. Sci. Eng.*, in press.
14. J. H. Renken, private communication, 1981.
15. J. H. Renken and K. G. Adams, "An Improved Capability for Solution of Photon Transport Problems by the Method of Discrete Ordinates," SC-RR-69-739, Sandia National Laboratories, Albuquerque, N.M., 1969.
16. H. Brockmann, *Nucl. Sci. Eng.* **77** (1981), 377.
17. M. Abramowitz and I. A. Stegun (Eds.), "Handbook of Mathematical Functions," Nat. Bur. Stds. Applied Math Series 55, 1964.
18. E. İnönü, *J. Math. Phys.* **11** (1970), 568.
19. J. C. P. Miller, *in* "British Association for the Advancement of Science, Bessel Functions, Part II. Functions of Integer Order, Mathematical Tables," Vol. X, p. xvii, Cambridge Univ. Press, London/New York, 1952.